On the extreme eigenvectors of certain Hermitian Toeplitz matrices

joint results with Albrecht Böttcher,
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Notation: Hermitian Toeplitz matrices

Let $a \in L^1(\mathbf{T}, \mathbf{R})$. Denote by a_k the Fourier coefficients of a:

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-ik\theta} \, d\theta.$$

Consider Toeplitz matrices $T_n(a)$, $n = 1, 2, 3, \ldots$, generated by a:

$$T_n(a) := (a_{j-k})_{j,k=1}^n = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \ldots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \ldots & a_{-n+2} \\ a_2 & a_1 & a_0 & \ldots & a_{-n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_0 \end{pmatrix}$$

Notation: eigenvalues and eigenvectors

Denote by $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ the eigenvalues of $T_n(a)$ in the increasing order:

$$\lambda_1^{(n)} \le \lambda_2^{(n)} \le \ldots \le \lambda_n^{(n)}$$

and by $v_1^{(n)}, \ldots, v_n^{(n)}$ the corresponding normalized eigenvectors:

$$T_n(a) v_k^{(n)} = \lambda_k^{(n)} v_k^{(n)}, \qquad \|v_k^{(n)}\|_2 = 1.$$

Under some assumptions on the symbol, the eigenvalues are simple. Therefore every normalized eigenvector is defined uniquely up to unitary multiplier τ, $|\tau| = 1$.

"Minimal distance" between normalized vectors

$$\varrho(u, v) := \min_{|\tau|=1} \|\tau u - v\|_2 = \left\| \frac{|\langle u, v\rangle|}{\langle u, v\rangle} u - v \right\|_2 .$$

Tridiagonal real symmetric Toeplitz matrices

As a model example, consider real three-term symbols:

$$a(t) = -t + 2 - t^{-1}$$

In this case the function $g(x) := a(e^{ix})$ is

$$g(x) = 4\sin^2\frac{x}{2},$$

and the Toeplitz matices are tridiagonal:

$$T_5(-t + 2 - t^{-1}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The eigenvalues and eigenvectors of such matrices are well known.

Eigenvectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $4\sin^2\frac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

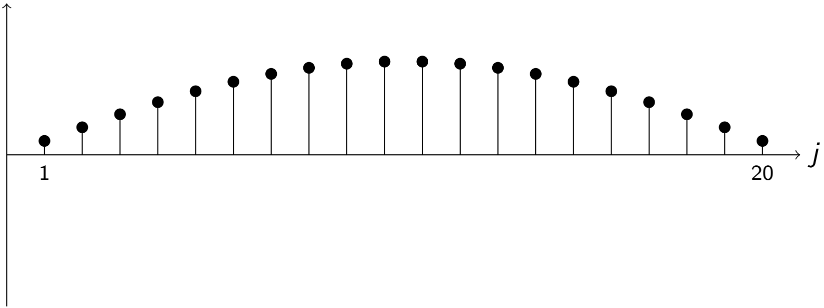
$$x_k^{(n)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{kj\pi}{n+1}\right)_{j=1}^{n}.$$

Eigenvectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $\quad 4\sin^2\dfrac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

$$x_1^{(20)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{j\pi}{21}\right)_{j=1}^{20}.$$

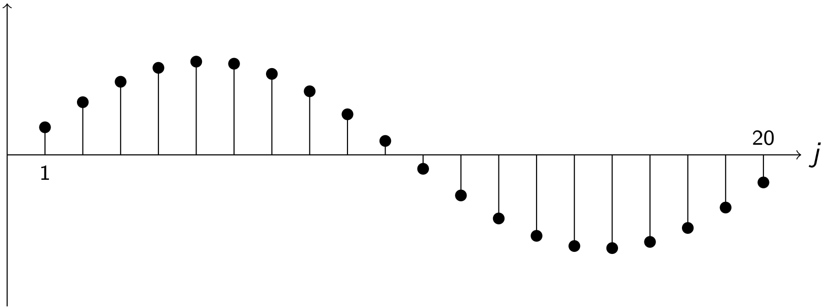


Eigenvectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $\quad 4\sin^2\dfrac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

$$x_2^{(20)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{2j\pi}{21}\right)_{j=1}^{20}.$$

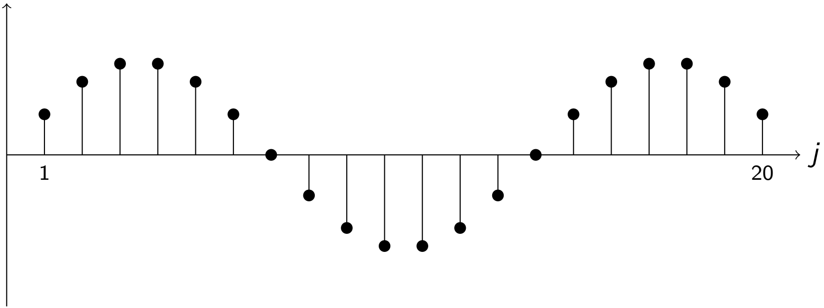


Eigenvectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $\quad 4\sin^2\dfrac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

$$x_3^{(20)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{3j\pi}{21}\right)_{j=1}^{20}.$$

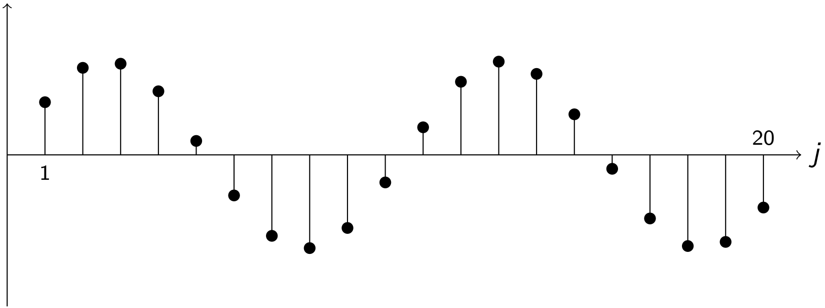


Eigenvectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t + 2 - t^{-1})$ is $\quad 4\sin^2 \dfrac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

$$x_4^{(20)} = \left(\sqrt{\frac{2}{\pi}} \sin \frac{4j\pi}{21}\right)_{j=1}^{20} .$$

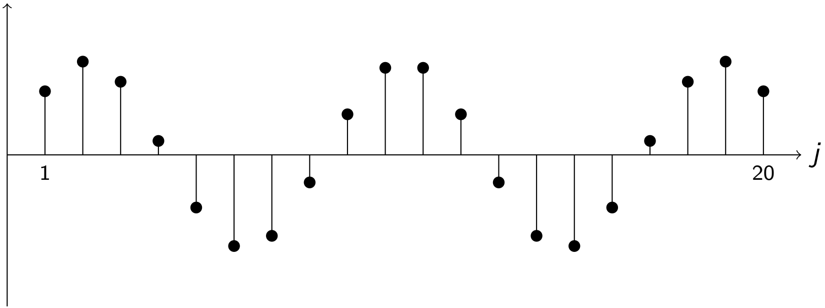


Eigenvectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $\quad 4\sin^2\dfrac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

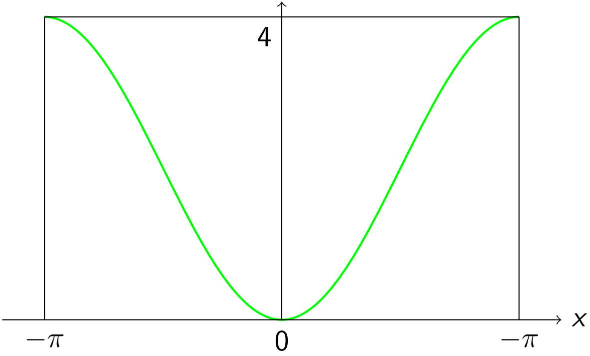
$$x_5^{(20)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{5j\pi}{21}\right)_{j=1}^{20}.$$



Important properties of the model generating function

The function $g(x) = 4\sin^2\frac{x}{2}$ has the following properties:

- g has only one minimum;
- The minimum is reached at the point 0;
- The minimum is of the second order.



Main message of the talk

Main assumptions

Suppose that the generating function is real-valued,
reaches its minimal value only at the point 0,
is sufficiently smooth near the point 0,
and the point 0 is a minimum of the second order.

Result

Under some additional assumptions of the symbol,

$$\forall j \qquad \lim_{n\to\infty} \varrho(v_j^{(n)}, x_j^{(n)}) = 0 \tag{1}$$

where $x_j^{(n)}$ are the eigenvectors of the tridiagonal Toeplitz matrices.

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Integrable symbols

Theorem

Let $a \in L_1(\mathbf{T}, \mathbf{R})$, $\operatorname{ess\,inf} a = m$, $\exists A > 0, \ B > 0, \ \sigma > 0$*:*

$$a(x) \geq m + Ax^2 \quad \forall x \in (-\pi, \pi); \tag{2}$$

$$a(x) \leq m + Ax^2 + Bx^4 \quad \forall x \in (-\sigma, \sigma). \tag{3}$$

Then for all $k = 1, 2, 3, \ldots$, *all* $n \geq N_k$, *and some* $\gamma > 0$

$$0 \leq \lambda_k^{(n)} - \left(m + 4A \sin^2 \frac{k\pi}{2(n+1)} \right) \leq \frac{\gamma k^3}{(n+1)^3} \tag{4}$$

and for some $\beta_k > 0$

$$\varrho \left(v_k^{(n)}, x_k^{(n)} \right) \leq \frac{\beta_k}{\sqrt{n+1}}. \tag{5}$$

The proof is based on estimates for the Rayleigh quotients.

Example with nonbounded integrable symbol

$$a(e^{ix}) = \left|1+e^{ix}\right|^{-2\alpha} = 2^{-2\alpha}\left(\cos\frac{x}{2}\right)^{-2\alpha}.$$

In this case $m = 2^{-2\alpha}$, $A = 4\alpha 2^{-2\alpha-2}$.

Consider error terms for eigenvalues and eigenvectors:

$$X_k^{(n)} := \lambda_k^{(n)} - m - 4A\sin^2\frac{k\pi}{2(n+1)}, \qquad Y_k^{(n)} := \varrho\left(v_k^{(n)}, x_k^{(n)}\right).$$

Numerical experiments (for $\alpha = 1/4$) show that

$$X_k^{(n)} = O\left(\frac{k^2}{n^3}\right), \qquad Y_k^{(n)} = O\left(\frac{k}{n}\right).$$

The theorem says only that

$$X_k^{(n)} = O\left(\frac{k^3}{n^3}\right), \qquad Y_k^{(n)} = O\left(\frac{\beta_k}{\sqrt{n}}\right).$$

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Instruments: formulas by Widom and Trench

Notation: complex roots of a polynomial symbol

Let a be a Laurent polinomial: $a(z) = \sum_{k=-q}^{p} a_k z^k \quad (a_p \neq 0,\ a_{-q} \neq 0)$.

Denote by $z_1(a), \ldots, z_{p+q}(a)$ the roots of the polynomial $z^q a(z)$.

Formulas by Widom and Trench

H. Widom (1958):

formulas for the determinants $\det T_n(a)$ in terms of $z_k(a)$.

W. F. Trench (1985):

formulas for the elements of inverse matrices $T_n^{-1}(a)$ in terms of $z_k(a)$.

These formulas are especially simple when the roots $z_1(a), \ldots, z_{p+q}(a)$ are pairwise distinct.

Assumptions on the symbol

- The symbol is a non-constant Laurent polynomial:

$$a(t) = \sum_{k=-r}^{r} a_k t^k, \qquad r \geq 1, \quad a_r \neq 0, \quad a_{-r} \neq 0.$$

- The symbol is real-valued on $\mathbf{T}$, i.e. $\overline{a_k} = a_{-k}$ for all k.
- $a(\mathbf{T}) = [0, M], \quad a(1) = 0, \quad a(e^{i\varphi_0}) = 0$ for some $\varphi_0 \in (0, 2\pi)$.
- $g(x) := a(e^{ix})$ is strictly increasing on $[0, \varphi_0]$, strictly decreasing on $[\varphi_0, 2\pi]$, $g''(0) \neq 0$ and $g''(\varphi_0) \neq 0$.
- Technical assumption: for each $\lambda \in (0, M)$ the roots of $a(z) - \lambda$ lying in $\mathbf{C} \setminus \mathbf{T}$ are pairwise distinct.

Auxiliary functions

For every $\lambda \in (0, M)$, the roots of $a(z) - \lambda$ can be written as

$$u_1(\lambda), \ldots, u_{r-1}(\lambda), \qquad e^{i\varphi_1(\lambda)},\ e^{i\varphi_2(\lambda)}, \qquad \frac{1}{\overline{u_1(\lambda)}}, \ldots, \frac{1}{\overline{u_{r-1}(\lambda)}},$$

where $\quad 0 < \varphi_1(\lambda) < \varphi_0, \quad \varphi_0 - 2\pi < \varphi_2(\lambda) < 0, \quad |u_k(\lambda)| \geq 1 + \delta_0.$

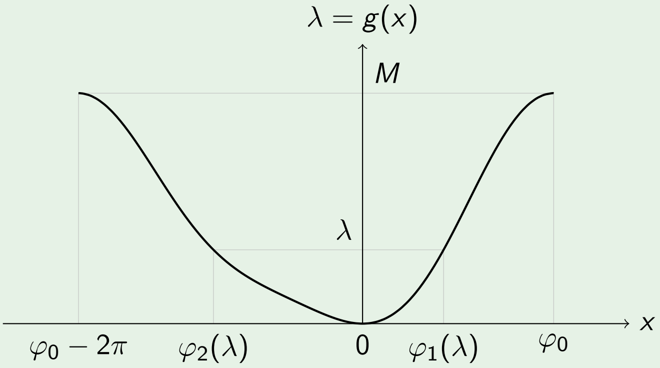
Put

$$h_\lambda(z) = \prod_{k=1}^{r-1} \left(1 - \frac{z}{u_k(z)}\right).$$

Function φ

$$\varphi(\lambda) := \frac{\varphi_1(\lambda) - \varphi_2(\lambda)}{2} = \frac{1}{2}\,\mathrm{mes}\{x \in [0, 2\pi) \colon g(x) \le \lambda\}.$$

Example ($g(x) = 16\sin^2\frac{x}{2} + 16\sin^3\frac{x}{2}\cos\frac{x}{2}$)

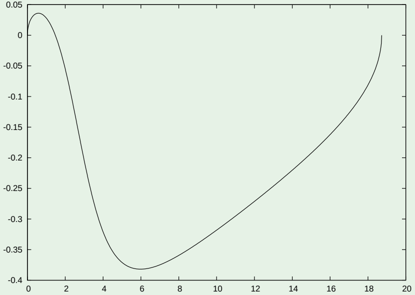


Function θ

θ is defined as the continuous argument of the function

$$\lambda \to h_\lambda(e^{i\varphi_1(\lambda)})/h_\lambda(e^{i\varphi_2(\lambda)}).$$

Example $\left(g(x) = 16\sin^2\frac{x}{2} + 16\sin^3\frac{x}{2}\cos\frac{x}{2}\right)$



Asymptotical equation for the eigenvalues

Theorem

The solution $\lambda_{j,*}^{(n)}$ *of the equation*

$$(n+1)\varphi(\lambda)+\theta(\lambda)=j\pi$$

is exponentially close to $\lambda_j^{(n)}$ *as* $n\to\infty$*:*

$$|\lambda_j^{(n)}-\lambda_{j,*}^{(n)}|\le Ke^{-\delta n}.$$

The solution of $(*)$ can be computed using the fixed point method:

$$\lambda_{j,0}^{(n)}:=\varphi^{-1}\left(\frac{j\pi}{n+1}\right),\qquad \lambda_{j,k}^{(n)}:=\varphi^{-1}\left(\frac{j\pi-\theta(\lambda_{j,k-1}^{(n)})}{n+1}\right).$$

Formulas for the eigenvectors (symmetric case)

Intoduce the vectors $y_j^{(n)}$ with the following coordinates:

$$y_{j,m}^{(n)} := \sin\left(m\varphi(\lambda) + \frac{\theta(\lambda)}{2}\right) - \sum_{k=1}^{r-1} Q_k(\lambda)\left(\frac{1}{u_k(\lambda)^m} + \frac{(-1)^{j+1}}{u_k(\lambda)^{n+1-m}}\right),$$

where $$Q_k(\lambda) = \frac{|h_\lambda(e^{i\varphi(\lambda)})|\sin\varphi(\lambda)}{(u_k(\lambda) - e^{i\varphi(\lambda)})(u_k(\lambda) - e^{i\varphi(\lambda)})h'_\lambda(u_k(\lambda))}.$$

Theorem

The following asymptotical formula for $v_j^{(m)}$ holds as $n \to \infty$:

$$v_j^{(n)} = \frac{\tau_j^{(n)}}{\|y_j^{(m)}\|} \cdot y_j^{(m)} + O(e^{-n\delta}) \qquad (1 \le j \le n),$$

where $|\tau_j^{(n)}| = 1$ and the constant in O depends only on the symbol.

First eigenvectors (symmetric case)

Theorem

$$\rho\left(v_j^{(n)}, x_j^{(n)}\right) = O\left(\frac{j}{n}\right), \qquad (*)$$

where $x_j^{(n)}$ are the eigenvectors of tridiagonal symmetric Toeplitz matrices.

Example ($g(x) = 4\sin^2\frac{x}{2} + 16\sin^4\frac{x}{2}$)

The values of $\rho\left(v_j^{(n)}, x_j^{(n)}\right)$ multiplied by $\frac{n+1}{j}$:

	$n = 100$	$n = 1000$	$n = 10000$
$j = 1$	1.212	1.273	1.279
$j = 2$	1.083	1.155	1.162
$j = 3$	1.052	1.132	1.139

So, in this example the order of the error term in $(*)$ is exact.

The results of this talk are published in the articles:

A. Böttcher, S. M. Grudsky, E. A. Maksimenko, and J. Unterberger, *The first order asymptotics of the extreme eigenvectors of certain Hermitian Toeplitz matrices*, Integral Equations Operator Theory, vol. 63 (2009), N 2, 165–180. http://www.springerlink.com/content/wkl4x567g855x670/

A. Böttcher, S. M. Grudsky, and E. A. Maksimenko, *On the structure of the eigenvectors of large Hermitian Toeplitz band matrices*, to appear. Preprint: http://www.mathematik.tu-chemnitz.de/preprint/2009/PREPRINT˙05.html

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