Dynamics of Toeplitz operators on weighted Bergman spaces

S.Grudsky

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This report is made on the basis of the joint works with
A.Karapetyants and N.Vasilevski.

$$\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}, \quad L_2(\mathbb{D}, d\mu_\lambda)$$

$$\|f\|_{L_2(\mathbb{D},d\mu_\lambda)} = \left(\int_{\mathbb{D}} |f(z)|^2 d\mu_\lambda(z)\right)^{1/2}$$

$$d\mu_\lambda(z) = (\lambda+1)(1-|z|^2)^\lambda \frac{1}{\pi} dv(z), \quad \lambda > -1, \tag{1}$$

where $dv(z) = dxdy$ is the Euclidian area element.

$\mathcal{A}^2_\lambda(\mathbb{D})(\in L_2(\mathbb{D}, d\mu_\lambda))$ is weight Bergman space of analytic functions

$$(\mathcal{B}^{(\lambda)}_{\mathbb{D}} f)(z) := \int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\bar{\zeta})^{\lambda+1}} d\mu_\lambda, \quad z \in \mathbb{D} \tag{2}$$

$\mathcal{B}^{(\lambda)}_{\mathbb{D}} : L_2(\mathbb{D}, d\mu_\lambda) \to \mathcal{A}^2_\lambda(\mathbb{D})$ is Bergman Projector on unit disk

$T^{(\lambda)}_a = \mathcal{B}^{(\lambda)}_{\mathbb{D}} a \mathcal{B}^{(\lambda)}_{\mathbb{D}} : \mathcal{A}^2_\lambda(\mathbb{D}) \to \mathcal{A}^2_\lambda(\mathbb{D})$ is Toeplitz-Bergman operator with symbol $a = a(z)(\in L_1(\mathbb{D}))$

DYNAMICS PROPERTIES OF THE TOEPLITZ OPERATORS ON THE WEIGHTED BERGMAN SPACE

S. Grudsky

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Toronto, August 2003

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The main theme: what happens to properties of Toeplitz-Bergman operators when the weight parameter λ varies, especially if $\lambda \to \infty$?

Motivation (in particular): Berezin quantization procedure

$$T_a = \{T_a^{(h)}\}, \quad h \in (0,1) \quad \left(h = \frac{1}{\lambda + 2}\right),$$

$$\lim_{h \to 0} \widetilde{a}_h = a \tag{3}$$

where $\widetilde{a}_h$ is Wick symbol.

Our goals are investigations boundedness, compactness and spectrum of Toeplitz-Bergman operator depending on λ.

Classes of symbols

1. Radial symbols on unit disk: $a = a(|z|)$, $z \in \mathbb{D}$.

2. Symbols dependent only on $y = \operatorname{Im} z$ on upper half-plane:
$a = a(y)$, $z \in \Pi = \{z = x + iy : x \in \mathbb{R}, y > 0\}$.

3. Symbols dependent on $\theta = \arg z$ on upper half-plane:
$a = a(\theta)$, $z \in \Pi$, $\theta \in (0, \pi)$.

Theorem 1 (Vasilevski, 2000) *C^*-algebras generated by Toeplitz-Bergman operators with symbols from* **1**, **2** *or* **3** *are commutative for each $\lambda > -1$.*

Theorem 2 *Toeplitz-Bergman operators T_a with radial symbols $a = a(|z|)$ from $L_1(0,1)$ (acting on $\mathcal{A}^2_\lambda(\mathbb{D})$) is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$ (acting on $l_2(\mathbb{Z}_+)$) where sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}_{n\in\mathbb{Z}_+}$ is given by*

$$\gamma_{a,\lambda}(n) = \frac{1}{B(n+1,\lambda+1)} \int_0^1 a(\sqrt{r})(1-r)^\lambda r^n dr, \quad n \in \mathbb{Z}_+. \tag{4}$$

Toeplitz-Bergman Operators on Half-plane

$$f \in L_2(\Pi, d\mu_\lambda) \iff \|f\|_{L_2(\Pi,d\mu_\lambda)} = \left(\int_\Pi |f(z)|^2 d\mu_\lambda(z)\right)^{1/2} < \infty$$

where

$$\mu_\lambda(z) = (\lambda+1)(2\operatorname{Im} z)^\lambda \frac{1}{\pi} dxdy. \tag{5}$$

$\mathcal{A}^2_\lambda(\Pi)(\subset L_2(\Pi, d\mu_\lambda))$ is weight Bergman space of analytic functions on half-plane.

$$(\mathcal{B}_\Pi f)(z) = \frac{\lambda+1}{\pi} \int_\Pi f(\zeta) \left(\frac{\zeta - \overline{\zeta}}{z - \zeta}\right)^{\lambda+1} \frac{dxdy}{(2\operatorname{Im}\zeta)^2} \tag{6}$$

$\mathcal{B}_\Pi : L_2(\Pi, d\mu_\lambda) \to \mathcal{A}^2_\lambda(\Pi)$ is Bergman Projector on half-plane.

$T_a^{(\lambda)} := \mathcal{B}_\Pi^{(\lambda)} a \mathcal{B}_\Pi^{(\lambda)} : \mathcal{A}^2_\lambda(\Pi) \to \mathcal{A}^2_\lambda(\Pi)$ is Bergman-Toeplitz operator.

Theorem 3 *Toeplitz-Bergman operator T_a with symbol $a = a(y)$ from $L_1(\mathbb{R}_+, 0)$ (acting on $\mathcal{A}^2_\lambda(\Pi)$) is unitary equivalent to the multiplication operator $\gamma_{a,\lambda}I$ (acting on $L_2(\mathbb{R}_+)$) where function $\gamma_{a,\lambda} = \gamma_{a,\lambda}(x)$ is given by*

$$\gamma_{a,\lambda}(x) = \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty a(t/2) t^\lambda e^{-xt} dt. \tag{7}$$

$$(a(y) \in L_1(\mathbb{R}_+, 0) \iff a(y)e^{-\varepsilon y} \in L_1(\mathbb{R}_+) \text{ for any } \varepsilon > 0)$$

Theorem 4 *Toeplitz-Bergman operator T_a with symbol $a = a(\theta)$ from $L_1(0, \pi)$ (acting on $\mathcal{A}^2_\lambda(\Pi)$) is unitary equivalent to multiplication operator $\gamma_{a,\lambda}I$ (acting on $L_2(\mathbb{R})$) where function $\gamma_{a,\lambda}(\xi)$ is given by*

$$\gamma_{a,\lambda}(\xi) = \frac{(\lambda+1)2^\lambda e^{\pi\xi}}{\pi} \cdot \frac{\left|\Gamma\left(\frac{\lambda+2}{2} + i\xi\right)\right|^2}{\Gamma(\lambda+2)} \int_0^\pi a(\theta) e^{-2\xi\theta} \sin^\lambda \theta d\theta, \ \xi \in \mathbb{R} \tag{8}$$

Boundedness and Compactness Properties (Radial case)

$$B_{a,\lambda_0}^{(1)}(s) = \int_s^1 a(\sqrt{r})(1-r)^{\lambda_0}dr; \quad B_{a,\lambda_0}^{(j)}(s) = \int_s^1 B_{a,\lambda_0}^{(j-1)}(r)dr, \tag{9}$$

$j = 2, 3, \ldots, \quad \lambda_0 \geq 0$

Theorem 5 *If there exist $j \in \mathbb{N}$ and $\lambda_0 \geq 0$ such that*

$$B_{a,\lambda_0}^{(j)}(r) = O((1-r)^{j+\lambda_0}), \quad r \to 1, \tag{10}$$

then the Toeplitz-Bergman operator $T_a^{(\lambda)}$ is bounded on each $\mathcal{A}_\lambda^2(\mathbb{D})$ with $\lambda \geq 0$.

If for some $j \in \mathbb{N}$ and $\lambda_0 \geq 0$

$$B_{a,\lambda_0}^{(j)}(r) = o((1-r)^{j+\lambda_0}), \quad r \to 1, \tag{11}$$

then the operator $T_a^{(\lambda)}$ is compact on each $\mathcal{A}_\lambda^2(\mathbb{D})$ with $\lambda \geq 0$.

Example 1 Unbounded symbol

$$a(r) = (1-r^2)^{-\beta} \sin(1-r^2)^{-\alpha} \tag{12}$$

where $\alpha > 0$ and $\beta \in (0, 1)$.

Theorem 5 $\implies$ $T_a^{(\lambda)}$ is bounded and compact for $\lambda \geq 0$.

Theorem 6 *Let either* $a(r) \geq 0$, *or* $B^{(j)}_{a,\lambda_0}(r) \geq 0$ *for a certain* $j \in \mathbb{N}$ *and* λ_0. *Then the conditions (10), (11) are also necessary for the boundedness and compactness of the corresponding Toeplitz operator* $T_a^{(\lambda)}$ *on* $\mathcal{A}^2_\lambda(\mathbb{D})$ *with* $\lambda \geq 0$, *respectively.*

Corollary 1 *If* $a(r) \geq 0$, *and* $\lim_{\varepsilon\to 0} \inf_{r\in[1-\varepsilon,1]} a(r) = +\infty$ *then the Toeplitz operator* $T_a^{(\lambda)}$ *is unbounded on each* $\mathcal{A}^2_\lambda(\mathbb{D})$, $\lambda \geq 0$.

Corollary 2 *Let* $a(\sqrt{r}) \in L_1(0,1)$, *and let* $a(r) \geq 0$, *or* $B^{(j)}_{a,\lambda_0}(r) \geq 0$ *for some* $j \in \mathbb{N}$. *Then the Toeplitz operator* $T_a^{(\lambda)}$ *is bounded (compact), or unbounded (not compact) on each* $\mathcal{A}^2_\lambda(\mathbb{D})$ *simultaneously.*

Boundedness and Compactness in Dependence on $\lambda \in [0, \infty)$

Theorem 7 *The following statements hold:*

(i) if for any $\lambda_0 > 0$, the sequence γ_{a,λ_0} is bounded, then the sequence $\gamma_{a,\lambda}$ is bounded for all $\lambda \in [0, \lambda_0)$;

(ii) if for any $\lambda_0 > 0$, $\lim_{n\to\infty} \gamma_{a,\lambda_0}(n) = 0$, then $\lim_{n\to\infty} \gamma_{a,\lambda}(n) = 0$ for all $\lambda \in [0, \lambda_0)$.

$$B(a) = \{\lambda \in [0, \infty) : T_a^{(\lambda)} \text{ is bounded}\}$$

$$K(a) = \{\lambda \in [0, \infty) : T_a^{(\lambda)} \text{ is compact}\}$$

(i) $[0, \infty)$ (ii) $[0, \lambda_0)$ (iii) $[0, \lambda_0]$

$$\gamma(n) = e^{\frac{i}{5\pi} \ln^2(n+1)} \ln^{-\nu}(n+1) \ln^{\beta} \ln(n+1) \tag{13}$$

There exists $a_{\nu,\beta}(r)$ ($\in L_1(0,1)$) such that $\gamma_{a_{\nu,\beta}}(r) = \gamma(n)$.

Theorem 8 *Let $0 < \nu < 1$. Then*

a) $B(a_{\nu,0}) = [0, \nu]$, $K(a_{\nu,0}) = [0, \nu)$, $\beta = 0$,

b) $B(a_{\nu,\beta}) = [0, \nu)$, $K(a_{\nu,\beta}) = [0, \nu)$, $\beta > 0$,

c) $B(a_{\nu,\beta}) = [0, \nu]$, $K(a_{\nu,\beta}) = [0, \nu]$, $\beta < 0$.

Algebra of Continuous Operators Functions

Case 3. *Symbols dependent on* $\theta = \arg z$ *on upper half-plane* $a = a(\theta), z \in \Pi$, $\theta \in (0, \pi)$.

Question. *What are conditions on symbols such that*

$$\gamma_{a,\lambda}(\xi) \in C(\bar{R})?$$

$$\lim_{\xi \to +\infty} \gamma_{a,\lambda}(\xi) = c_+ \quad \textit{and} \quad \lim_{\xi \to -\infty} \gamma_{a,\lambda}(\xi) = c_- \ (!)$$

Let $f \in C[0,1]$, *then there exists symbol* $a(\theta)$ *such that*

$$f\left(T_{\chi(0,\frac{\pi}{2})}\right) = T_a?$$

$$\gamma_{\chi(0,\frac{\pi}{2}),0} = \frac{1}{e^{-\pi\lambda}+1} \Rightarrow f\left(\frac{1}{e^{-\pi\lambda}+1}\right) = \gamma_{a,0}(\lambda)$$

For any L_1-symbol $a(\theta)$ we define the following averaging functions, which correspond to the endpoints of $[0,\pi]$,

$$C_a^{(1)}(\theta) = \int_0^\theta a(u)du, \qquad D_a^{(1)}(\theta) = \int_{\pi-\theta}^\pi a(u)du$$

and

$$C_a^{(p)}(\theta) = \int_0^\theta C_a^{(p-1)}(u)du,$$

$$D_a^{(p)}(\theta) = \int_{\pi-\theta}^\pi D_a^{(p-1)}(u)du,$$

for each $p = 2, 3, \ldots$.

Theorem 0.1. *Let $a(\theta) \in L_1(0,\pi)$ and for some $p, q \in \mathbb{N}$,*

$$\lim_{\theta\to 0} \theta^{-p} C_a^{(p)}(\theta) = c_p \ (\in \mathbb{C}) \qquad \text{and}$$

$$\lim_{\theta\to \pi} \theta^{-q}\, D_a^{(q)}(\theta) = d_q \ (\in \mathbb{C}). \qquad (0.1)$$

Then $\gamma_a(\lambda) \in C(\overline{\mathbb{R}})$.

EXAMPLE 0.2. Let

$$a(\theta) = \theta^{-\beta} \sin \theta^{-\alpha}, \text{ where } 0 \leq \beta < 1, \ \alpha > 0. \tag{0.2}$$

This symbol oscillates near 0, is bounded when $\beta = 0$, is unbounded for all $\beta \in (0,1)$. According to asymptotics calculations we have that

$$C_a^{(1)}(\theta) = \frac{\theta^{\alpha-\beta+1}}{\alpha} \cos \theta^{-\alpha} + O(\theta^{2\alpha-\beta+1}),$$

$$\text{when} \quad \theta \to 0. \tag{0.3}$$

Thus, if $\alpha > \beta$ then

$$\lim_{\theta \to 0} \theta^{-1} C_a^{(1)}(\theta) = 0,$$

and the first condition in (0.1) is satisfied for $p = 1$.

Further, if $\alpha \leq \beta$ we need to consider the averages of the higher order. Indeed, formula (0.3) implies that

$$C_a^{(2)}(\theta) = O(\theta^{2\alpha-\beta+2}), \qquad \text{when} \quad \theta \to 0$$

and, more generally, that

$$C_a^{(p)}(\theta) = O(\theta^{p\alpha-\beta+p}), \qquad \text{when} \quad \theta \to 0.$$

Thus for each $\alpha \leq \beta$ there is $p_0 \in \mathbb{N}$ such that $p_0\alpha > \beta$, and thus the first condition in (0.1) is satisfied for $p = p_0$.

That is, the Toeplitz operator T_a with symbol (0.2) satisfies of the conditions (!) for all admissible values of the parameters.

Given any $a(\theta) \in L_\infty(0,\pi)$, we introduce now two modified averaging functions which correspond to the endpoints of $[0,\pi]$

$$C'_a(\theta) = \frac{2}{1-e^{-2\theta}} \int_0^\theta a(u)\,du \qquad \text{and}$$

$$D'_a(\theta) = \frac{2}{1-e^{-2\theta}} \int_{\pi-\theta}^\pi a(u)\,du. \qquad (0.4)$$

We note that these functions are connected with the old averaging ones as follows

$$C'_a(\theta) = \frac{2}{1-e^{-2\theta}}\, C_a^{(1)}(\theta) \qquad \text{and}$$

$$D'_a(\theta) = \frac{2}{1-e^{-2\theta}}\, D_a^{(1)}(\theta).$$

Theorem 0.3. *Let $a(\theta) \in L_\infty(0,\pi)$. Then $\gamma_a(\lambda) \in C(\overline{\mathbb{R}})$ if and only if*

$$\gamma_{C'_a}(\lambda) \in C(\overline{\mathbb{R}}) \quad \text{and} \quad \gamma_{D'_a}(\lambda) \in C(\overline{\mathbb{R}}). \quad (0.5)$$

Shatten Classes

$$T_a^{(\lambda)} \in K_p(\lambda) \iff \|T_a^\lambda\|_{p,\lambda} = \left(\sum_{n=1}^{\infty} |\gamma_{a,\lambda}(n)|^p\right)^{1/p} < \infty,\ p \geq 1 \tag{14}$$

Theorem 9 *Let $a(\sqrt{r}) \in L_1(0,1)$ and let for some $j = 0, 1, \ldots,$ the function $B_{a,\lambda}^{(j)}(r)$ satisfy one of the following conditions*

$$\int_0^1 |B_{a,\lambda}^{(j)}(r)|(1-r)^{-(1+j+\frac{1}{p})}\, dr < \infty, \qquad p \geq 1,$$
$$\int_0^1 |B_{a,\lambda}^{(j)}(r)|^p(1-r)^{-(2+j-\varepsilon)}\, dr < \infty, \qquad p > 1,$$

where $\varepsilon > 0$ can be arbitrarily small. Then $T_a^{(\lambda)} \in K_p(\lambda)$.

Example 2 Let $a > 0$, $b > 2 + \frac{1}{p}$, $\varepsilon_n = \frac{n^{-b}}{2}$.

$$a(\sqrt{r}) = \begin{cases} n^a,\ r \in I_n = \left[1 - \frac{1}{n}, 1 - \frac{1}{n} + \varepsilon_n\right], \\ 0,\ r \in [0,1] \setminus \cup_{n=1}^{\infty} I_n. \end{cases}$$

Then $T_a \in K_p(\lambda)$ $(\lambda \geq 0)$.

Spectra of Toeplitz Operators with Continuous Symbols
$(a = a(\theta),\ \theta \in (0, \pi))$

Let E be a subset of $\mathbb{R}$ having $+\infty$ as a limit point (normally $E = (0, +\infty)$), and let for each $\lambda \in E$ there is a set $M_\lambda \subset \mathbb{C}$. Define the set M_∞ as the set of all $z \in \mathbb{C}$ for which there exists a sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$ there exists $\lambda_n \in E$ such that $z_n \in M_{\lambda_n}$,

(ii) $\lim_{n\to\infty} \lambda_n = +\infty$,

(iii) $z = \lim_{n\to\infty} z_n$.

We will write

$$M_\infty = \lim_{\lambda\to+\infty} M_\lambda,$$

and call M_∞ the (partial) limit set of a family $\{M_\lambda\}_{\lambda\in E}$ when $\lambda \to +\infty$.

$T_a^{(\lambda)} \ :\ \mathcal{A}_\lambda^2(\Pi) \ \to\ \mathcal{A}_\lambda^2(\Pi)$ is unitary equivalent to

$\gamma_{a,\lambda} I \ :\ L_2(\mathbb{R}) \ \to\ L_2(\mathbb{R})$

Thus $\operatorname{sp} T_a^\lambda = \overline{M_\lambda}(a)$ where $M_\lambda(a) := \operatorname{Range} \gamma_{a,\lambda}(\xi)$ $(\xi \in \mathbb{R})$.

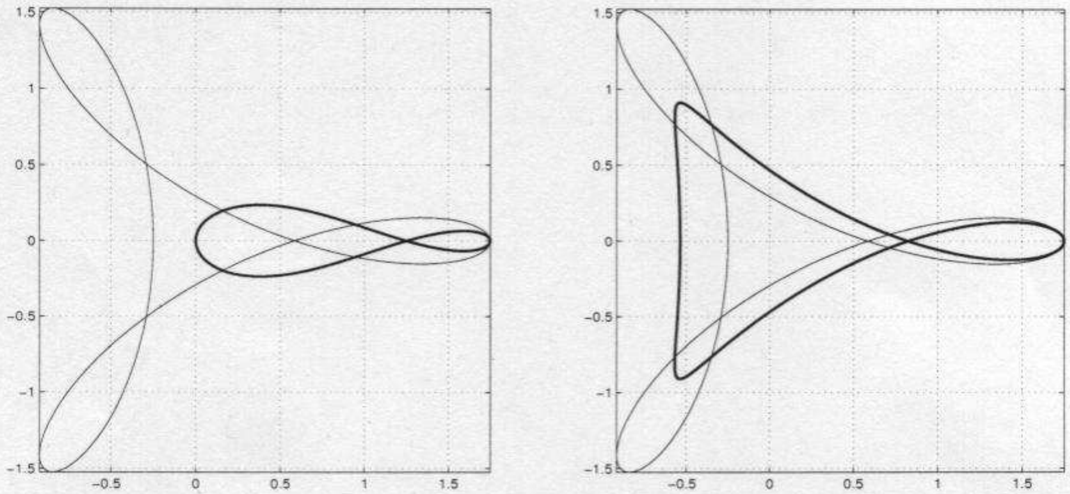
Theorem 10 *Let* $a = a(\theta) \in C[0, \pi]$. *Then*

$$\lim_{\lambda\to\infty} \operatorname{sp} T_a^{(\lambda)} = \operatorname{Range} a.$$

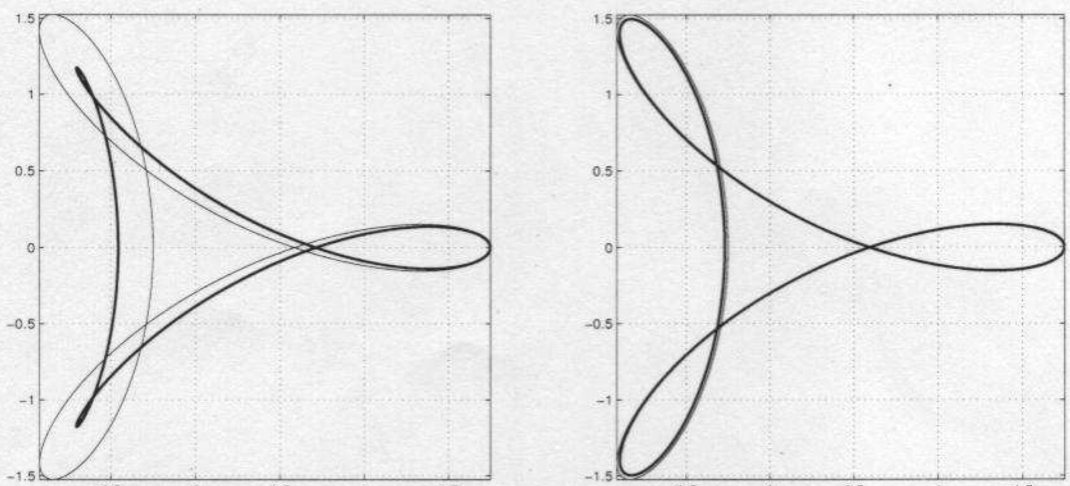
Example 3 (Hypocycloid)

$$a(\theta) = \frac{3}{4}e^{4i\theta} + e^{-2i\theta}, \quad \theta \in [0, \pi] \tag{15}$$

$\lambda = 0; 5; 12$ and 200



The function $\gamma_{a,\lambda}$ for $\lambda = 0$ and $\lambda = 5$.



The function $\gamma_{a,\lambda}$ for $\lambda = 12$ and $\lambda = 200$.

Piecewise Continuous symbols

$a = a(\theta),\ \theta \in [0, \pi))$

Let $a(\theta)$ be a piecewise continuous function having jumps on a finite set of points $\{\theta_j\}_{j=1}^m$ where

$$\theta_0 = 0 < \theta_1 < \theta_2 < \ldots < \theta_m < \pi = \theta_{m+1},$$

and $a(\theta_j \pm 0), j = 1, \ldots, m,$ exist. Introduce the sets

$$J_j(a) := \{z \in \mathbb{C} : z = a(\theta),\ \theta \in (\theta_j, \theta_{j+1})\}$$

where $j = 0, \ldots, m$, and let $I_j(a)$ be the segment with the end-points $a(\theta_j - 0)$ and $a(\theta_j + 0)$, $j = 1, 2, \ldots m$. We set

$$\widetilde{R}(a) = \left(\cup_{j=0}^m J_j(a)\right) \cup \left(\cup_{j=1}^m I_j(a)\right).$$

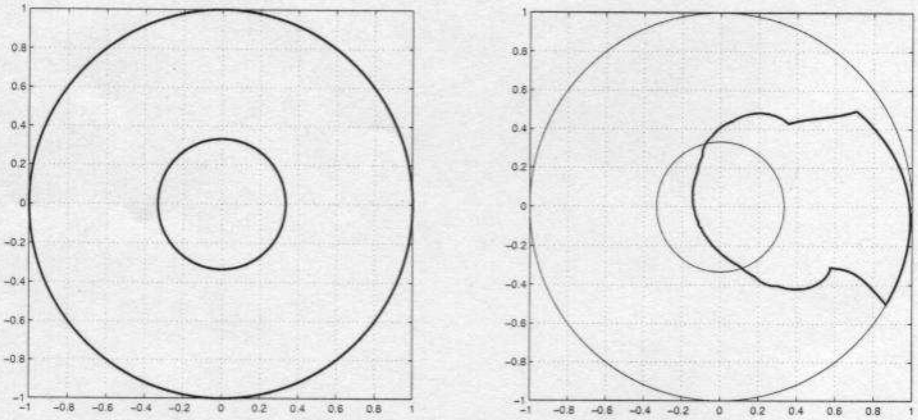
Theorem 11 *Let $a(\theta)$ be a piecewise continuous function. Then*

$$\lim_{\lambda\to\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) = \widetilde{R}(a).$$

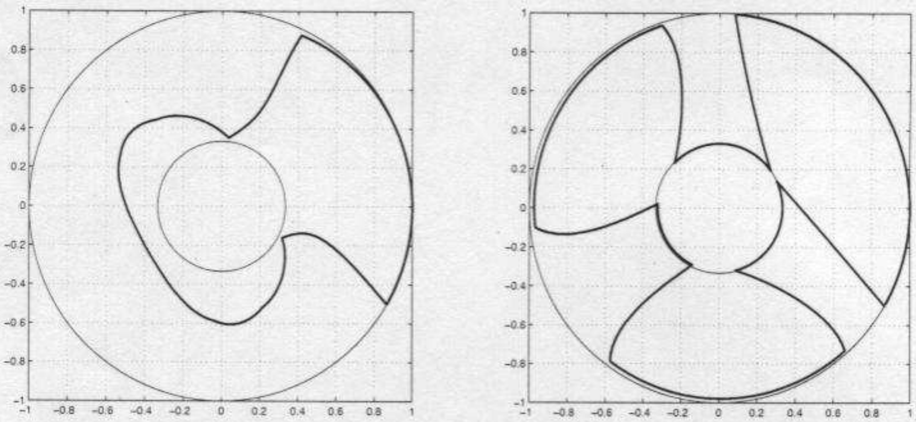
Example 4

$$a(\theta) = \begin{cases} \exp i\left[-\frac{\pi}{6} + \frac{2\pi}{3} \cdot \frac{7\theta}{\pi}\right], & \theta \in \left[0, \frac{\pi}{7}\right) \\ \frac{1}{3} \exp i\left[\frac{\pi}{6} + \frac{2\pi}{3} \cdot \left(\frac{7\theta}{\pi} - 1\right)\right], & \theta \in \left[\frac{\pi}{7}, \frac{2\pi}{7}\right) \\ \exp i\left[-\frac{\pi}{6} + \frac{2\pi}{3} \cdot \left(\frac{7\theta}{\pi} - 2\right)\right], & \theta \in \left[\frac{2\pi}{7}, \frac{3\pi}{7}\right) \\ \frac{1}{3} \exp i\left[-\frac{\pi}{6} + \frac{2\pi}{3} \cdot \left(\frac{7\theta}{\pi} - 3\right)\right], & \theta \in \left[\frac{3\pi}{7}, \frac{4\pi}{7}\right) \\ \exp i\left[-\frac{\pi}{6} + \frac{2\pi}{3} \cdot \left(\frac{7\theta}{\pi} - 4\right)\right], & \theta \in \left[\frac{4\pi}{7}, \frac{5\pi}{7}\right) \\ \frac{1}{3} \exp i\left[-\frac{\pi}{6} + \frac{2\pi}{3} \cdot \left(\frac{7\theta}{\pi} - 5\right)\right], & \theta \in \left[\frac{5\pi}{7}, \frac{6\pi}{7}\right) \\ \exp\left(-i\frac{\pi}{6}\right), & \theta \in \left[\frac{6\pi}{7}, \pi\right] \end{cases}$$

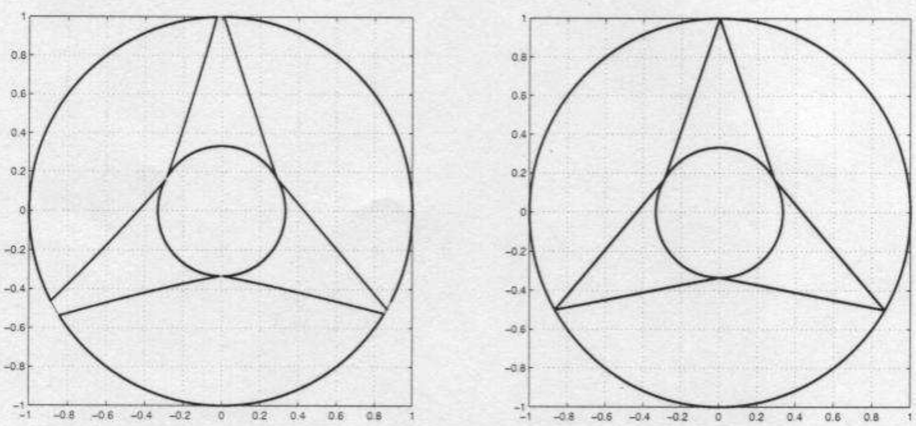
$\lambda = 1; 10; 70; 500$



The symbol $a(\theta)$ and the function $\gamma_{a,\lambda}$ for $\lambda = 1$.



The function $\gamma_{a,\lambda}$ for $\lambda = 10$ and $\lambda = 100$.



The function $\gamma_{a,\lambda}$ for $\lambda = 1000$ and the limit set $M_\infty(a)$.

Oscillating Symbols
$(a = a(y),\ y > 0)$

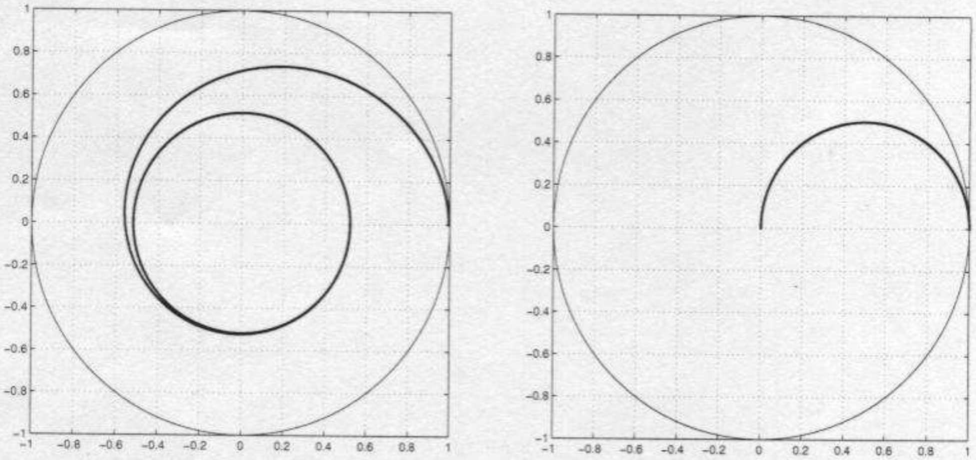
Theorem 12 (Strong oscillation) *Let* $a(y) = e^{2iy}$, *then* $\operatorname{Range} a = \mathbb{T}$ *and* $M_\infty(a) = \mathbb{D}$.

Theorem 13 (Slow oscillation) *Let* $a(y) = (2y)^i$, *then* $\operatorname{Range} a = \mathbb{T}$ *and* $M_\infty(a) = \mathbb{T}$.

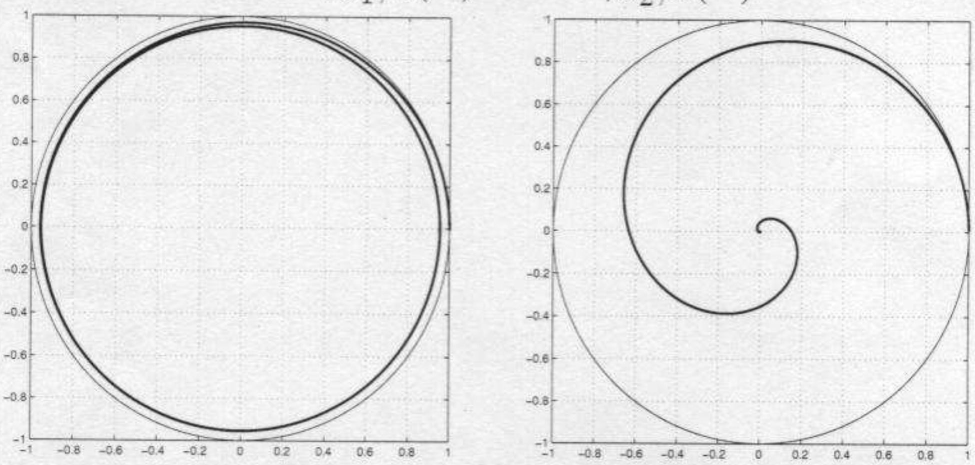
Example 5

$$a_1(y) = (1+2y)^i \quad \textit{and} \quad a_2(y) = e^{i2y}, \quad y \in [0, \infty).$$

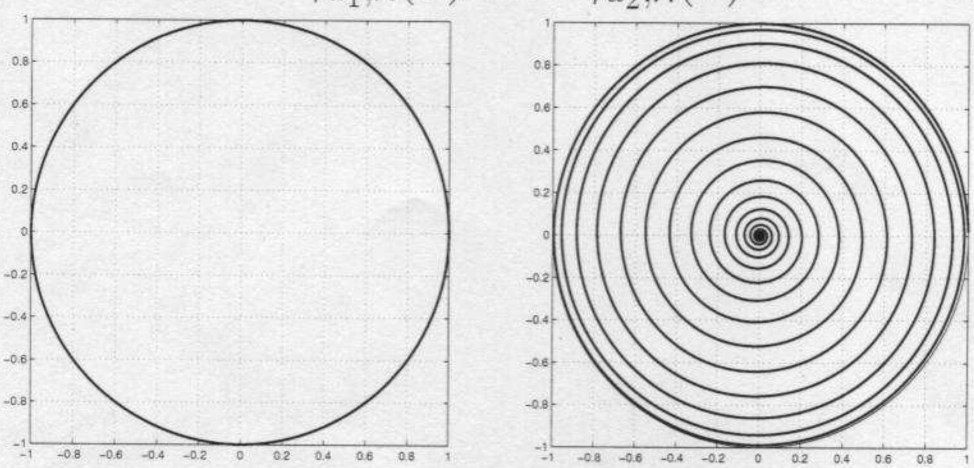
$$\lambda = 0; 10; 1000$$



The functions $\gamma_{a_1,\lambda}(x)$ and $\gamma_{a_2,\lambda}(x)$ for $\lambda = 0$.



The functions $\gamma_{a_1,\lambda}(x)$ and $\gamma_{a_2,\lambda}(x)$ for $\lambda = 10$.



The functions $\gamma_{a_1,\lambda}(x)$ and $\gamma_{a_2,\lambda}(x)$ for $\lambda = 1000$.

Unbounded Symbol (Radial Case)

Theorem 14 *Let* $a(\sqrt{r}) \in L_1(0,1) \cap C[0,1)$*. Then*

$$\operatorname{Range} a \subset M_\infty(a).$$

Theorem 15 *Let* $a(\sqrt{r}) \in L_1(0,1)$*. Then*

$$M_\infty(a) \subset \operatorname{conv}(\operatorname{essRange} a).$$

Example 6 Let $I_j := [1 - j^{-1} - j^{-3}, 1 - j^{-1}]$ and sequence $\{\theta_j\}_{j\in\mathbb{N}} \subset (0, 2\pi)$ with $\overline{\{\theta_j\}}_{j\in\mathbb{Z}_+} = [0, 2\pi]$. Consider

$$a(\sqrt{r}) = \begin{cases} je^{i\theta_j}, & r \in I_j, \\ 0, & r \in [0,1] \setminus \cup_{j=1}^{\infty} I_j. \end{cases}$$

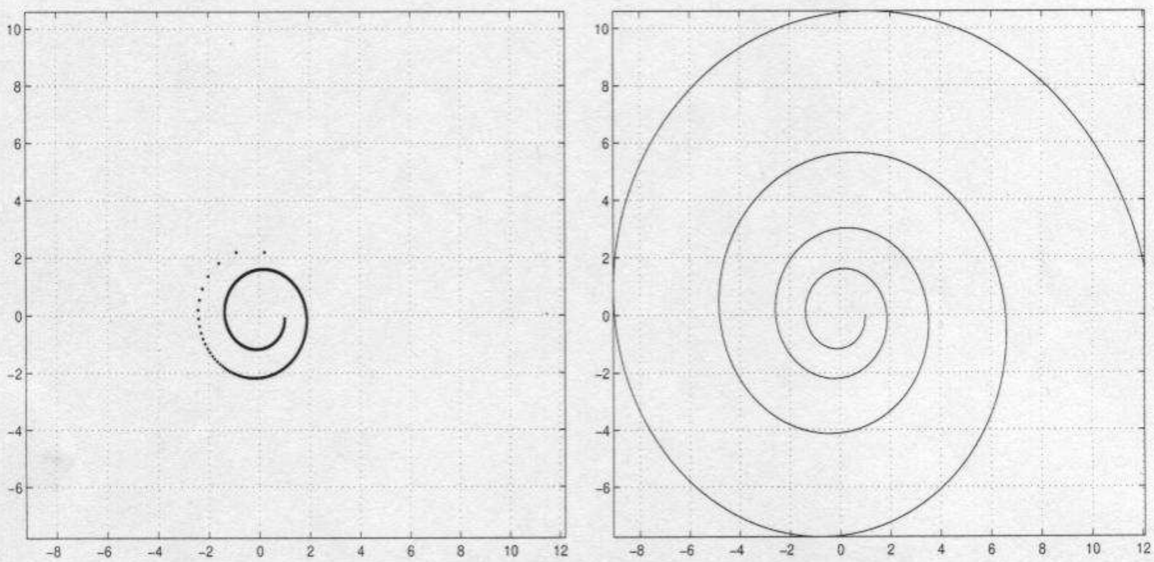
$$M_\infty(a) = \mathbb{C}$$

Example 7

$$a(\sqrt{r}) = r^{i-\alpha}, \qquad \alpha \in (0,1)$$

$$M_\infty(a) = \operatorname{Range} a$$

$$a(r) = r^{i-0.1}, \quad \lambda = 10^5$$



The sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}$ for $\lambda = 100000$ and the limit set $M_\infty(a)$.