# ASYMPTOTIC BEHAVIOR OF CONDITION NUMBERS OF FINITE TOEPLITZ MATRICES

**S. Grudsky**
(México, November 2004)

*This report is based on joint works with A. Böttcher*

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## Toeplitz Matrices

$$T(a) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

$$a(t) = \sum_{j=-\infty}^{\infty} a_j t^j, t \in \mathbb{T}$$

$$\mathbb{T} = \{t \in \mathbb{C} \mid |t| = 1\}$$

## Finite Toeplitz Matrices

$$T_n(a) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & & \dots & a_n \\ a_1 & a_0 & a_{-1} & & \dots & a_{-n+1} \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \end{pmatrix}$$

$$T(a) : l_p(\mathbb{Z}_+) \to l_p(\mathbb{Z}_+); ||X||_p =$$

$$= \left( \sum_{j=0}^{\infty} |X_j|^p \right)^{1/p}$$

$$T_n(a) : l_p(\mathbb{C}^{n+1}) \to l_p(\mathbb{C}^{n+1}); ||X||_p =$$

$$= \left( \sum_{j=0}^{n} |X_j|^p \right)^{1/p}$$

$$1 \leq p \leq \infty$$

## Condition number

$$\kappa_p(A) = \begin{cases} ||A||_p ||A^{-1}||_p & \text{, if } A \text{ is invertible} \\ \infty & \text{, if } A \text{ is not invertible} \end{cases}$$

## MAIN PROBLEM:

Behavior of $\kappa_p(T_n(a))$ if $n \to \infty$

## Spectral Theory of Toeplitz Operator

$$\operatorname{sp} A = \{\lambda \in \mathbb{C} | (A - \lambda I) \text{ is not invertible in } B\}$$

$$\operatorname{sp}_{ess} A = \{\lambda \in \mathbb{C} | (A - \lambda I) \text{ is not Fredholm in } B\}$$

$A$ is Fredholm $\Leftrightarrow$ $\operatorname{im} A = \overline{\operatorname{im} A}$, and
$\alpha := \dim \ker A < \infty$ and $\beta := \dim(B/\operatorname{im} A) < \infty$ and
$\operatorname{ind} A := \alpha - \beta$

**Theorem 1** *Let $a \in C(\mathbb{T})$, then*

**i)** *$T(a)$ is Fredholm on the space $l_p(\mathbb{Z}_+)$ if and only if*

$$\inf_{t \in \Pi} |a(t)| > 0 \tag{1}$$

*if condition (1) holds then* $\operatorname{ind} T(a) = -\text{wind } a(t)$

**ii)** *$T(a)$ is invertible if and only if the condition (1) holds and*

$$\text{wind } a(t) = 0 \tag{2}$$

**Corollary 1** *If $a \in C(\mathbb{T})$, then*

$$\operatorname{sp} T(a) = a(\boldsymbol{T}) \cup \big\{\lambda \in \boldsymbol{C} \setminus a(\boldsymbol{T}) : \\ \operatorname{wind}(a - \lambda) \neq 0\big\}.$$

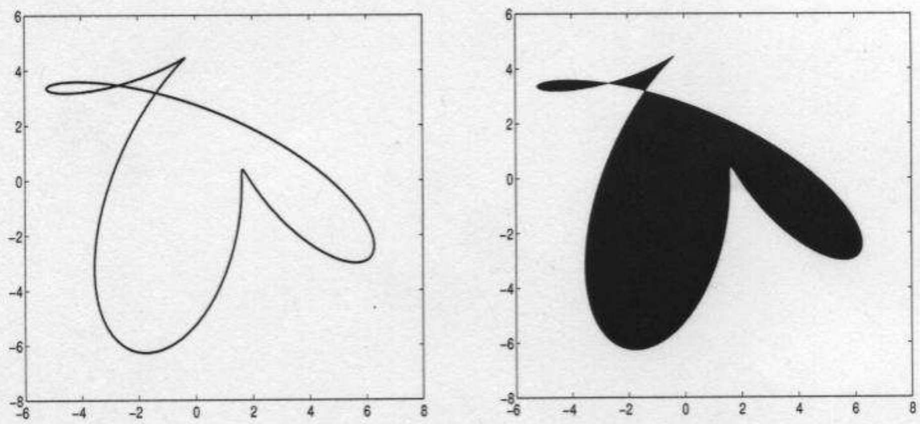


Figure 1: The set $\operatorname{sp}_{ess} T(a) = a(\mathbf{T})$ on the left and the set $\operatorname{sp} T(a)$ on the right.

## Conditions numbers of Toeplitz operators

$$\kappa_p(T(a)) =$$

$$\begin{cases} ||T(a)||_p||T^{-1}(a)||_p & \text{, if } a(t) \neq 0 \text{ and wind } a = 0 \\ \infty & \text{, otherwice} \end{cases}$$

**I** $T(a)$ is invertible: $a(t) \neq 0$ and wind $a = 0$

**II** $T(a)$ is Fredholm: $a(t) \neq 0$ and wind $a = 0$

**III** $T(a)$ is not Fredholm: $\inf a(t) = 0$

## I. *T(a)* is invertible

**Theorem 2 (Baxter, Gohrberg and Feldman)** *Let $a \in C(\mathbb{T})$ and $1 \leq p \leq \infty$. Then*

$$\limsup_{n\to\infty} ||T_n^{-1}(a)||_p < \infty \text{ if } T(a) \text{ is invertible}$$

$$\limsup_{n\to\infty} ||T_n^{-1}(a)|| = \infty \text{ if } T(a) \text{is not invertible}$$

$$||T_n(a)||_p \leq ||T(a)||_p \Rightarrow \limsup_{n\to\infty} \kappa_p(\mathbb{T}_n(a)) < \infty \text{ if } T(a) \text{is invertible}$$

## II. *T(a)* is Fredholm $(a(t) \neq 0)$

$$a(t) \in \mathcal{P}_{r,s} \Leftrightarrow a(t) = \sum_{j=-r}^{s} a_j t^j, \quad r, m > 0, \; a_{-r} \neq 0, \; a_s \neq 0$$

V

$$a(T) = t^{-r}(a_{-r} + a_{-r+1}t + \ldots + a_s t^{s+r})$$

$$a(t) = t^{-r} b_s \prod_{j=1}^{J}(t - \delta_j) \prod_{k=1}^{K}(t - \mu_k)$$

where $|\delta_j| < 1$ and $|\mu_k| > 1$

$$\delta = \max(|\delta_1|, \ldots, |\delta_J|), \quad \mu = \min(|\mu_1|, \ldots, |\mu_K|).$$

**Theorem 3** *Let $b$ be a Laurent polynomial and suppose wind $b \neq 0$. Let further $1 \leq p \leq \infty$. Then for every*

$$\alpha < \min\left(\log \frac{1}{\delta}, \log \mu\right)$$

*there is a constant $C_\alpha$ depending only on $\alpha$ (and $b, p$) such that*

$$\kappa_p(T_n(b)_p) \geq C_\alpha e^{\alpha n} \text{ for all } n \geq 1.$$

Figure 4.1 shows the norms $||T_n^{-1}(b-\lambda)||_2$ $(5 \leq n \leq 80)$ for $b(t) = t^{-2} + 0.75 \cdot t^{-1} + 0.65 \cdot t$ and $\lambda = -0.5,\ 0.82,\ 0.83 + 0.7i$ (top pictures and left picture in the middle) and for $b(t) = t^{-2} - 2\,t^{-1} + 1.25 \cdot t^3$ and $\lambda = -3.405,\ 1.48,\ 0.995 + 3i$ (right middle picture and bottom pictures). The curve $b(\mathbf{T})$ and the point $\lambda$ are indicated in the lower right corners of the pictures.

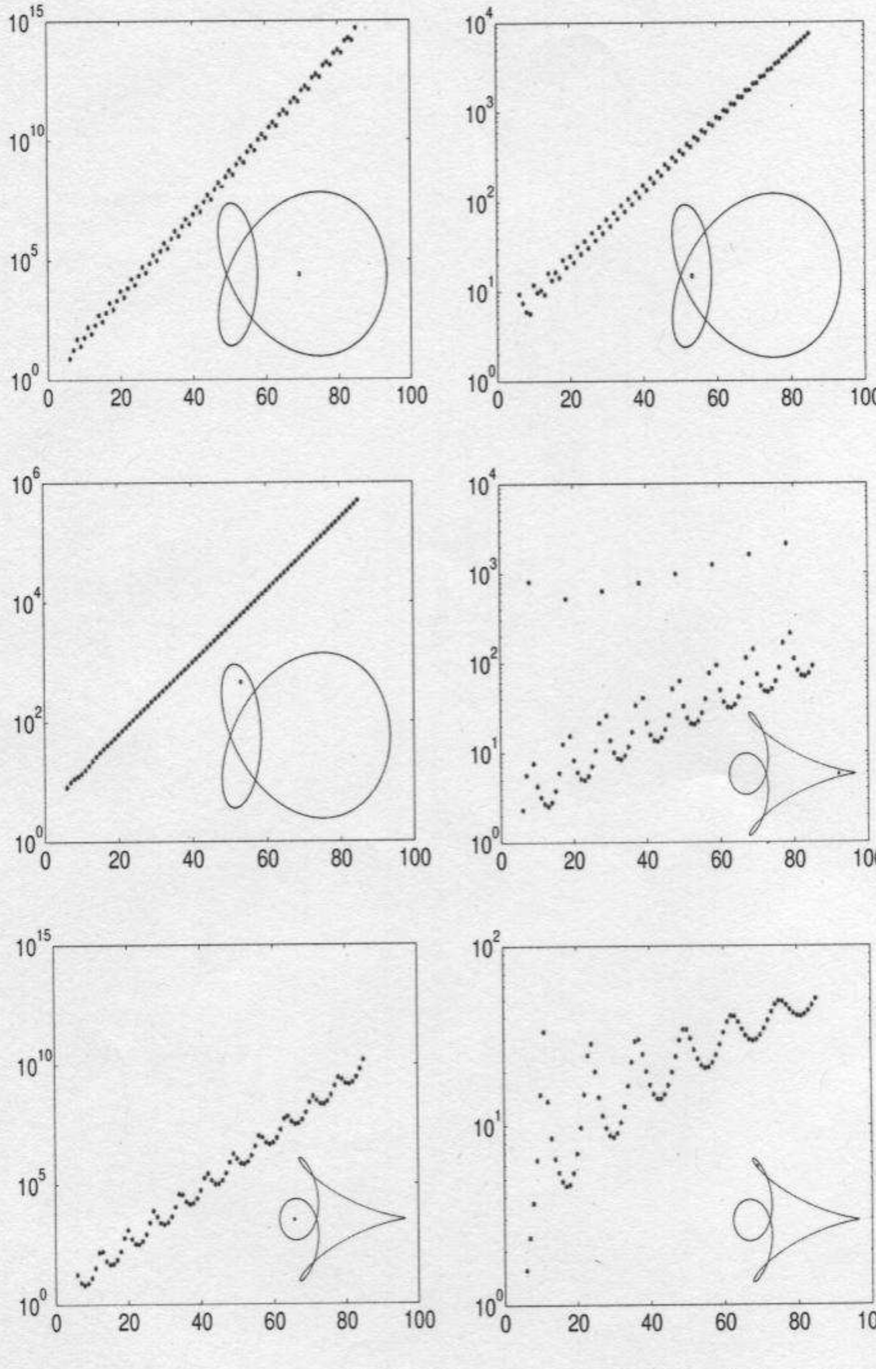


Figure 4.1: Norms $\|T_n^{-1}(b-\lambda)\|_2$ for two symbols $b$ and three $\lambda$'s.

## UPPER BONDS FOR CASE II

**Example 1** *$a(t) = t$. $T_n(t)$ is not invertible for arbitrary $n \in N$*

### *Generic case*

$$b(t) \in \mathcal{D} \Leftrightarrow b(t) = b_s t^{-r}(t - z_1)\dots(t - z_{r+s})(t \in \mathbb{T})$$
$$\text{where } b_s \neq 0 \text{ and } 0 < |z_1| < |z_2| < \dots < |z_{r+s}|$$

**Theorem 4** *Let $1 \leq p \leq \infty$. If $b \in \mathcal{D}$, then there are constants $\gamma \in (0, \infty)$ and $D_\gamma \in (0, \infty)$ depending only on $b$ and $p$ such that*

$$||\kappa_p(T_n(b)) \leq D_\gamma e^{\gamma n}$$

*for all $n \geq 1$.*

**Theorem 5** *Let $1 \leq p \leq \infty$ and let $\mathcal{E}$ be the set of all Laurent polynomials that have no zeros on $\mathbf{T}$ and whose winding number is nonzero.*

**(a)** *$\mathcal{E} \cap \mathcal{D}$ is a dense and open subset of the set $\mathcal{E}$ (with the uniform metric).*

**(b)** *If $b \in \mathcal{E} \cap \mathcal{D}$, then there are constants $C_1, C_2 \in (0, \infty)$ and $\gamma_1, \gamma_2 \in (0, \infty)$ depending only on $b$ and $p$ such that*

$$C_1 e^{\gamma_1 n} \leq \kappa_p(T_n(b)) \leq C_2 e^{\gamma_{2,n}}$$

*for all $n \geq 1$*

**Example 2** *Let*

$$T(b) = \begin{pmatrix} 0 & -4 & 0 & \dots \\ 1 & 0 & -4 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Then $b(t) = t - 4t^{-1} = t^{-1}(t-2)(t+2)$, which shows that wind $b = -1$.

$$|D_n(b)| = \frac{|2^{n+1} - (-2)^{n+1}|}{|2-(-2)|} = \begin{cases} 2^n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

$$C_1 e^{0.69n} \le \kappa_2(T_n(b)) \le C_2 e^{0.73n} \text{ for all even } n,$$

and we have

$$\kappa_2(T_n(b)) = \infty \text{ for all odd } n.$$

## Arbitrary Fast Growth

Pick $\alpha \in (0,1)$ and put

$$b(t) = t + \alpha^2 t^{-1} = t^{-1}(t + i\alpha)(t - i\alpha) \quad (t \in \mathbf{T}). \quad (3)$$

Since $b(e^{i\theta}) = (1+\alpha^2)\cos\theta + i(1-\alpha^2)\sin\theta$, we see

that $b(\mathbf{T})$ is an ellipse with the foci $-2\alpha$ and $2\alpha$.

If $\lambda \in (-2\alpha, 2\alpha)$, then $b - \lambda$ has no zeros on $\mathbf{T}$ and

$$\text{wind}(b - \lambda) \in \{-1, 1\}.$$

**Theorem 6** *Let $\varphi : \mathbf{N} \to \mathbf{N}$ be any function, for example, $\varphi(n) = \exp(n^n)$, and let $1 \le p \le \infty$. Then, with $b$ given by (3), there exists a $\lambda \in (-2\alpha, 2\alpha)$ such that $||T_{n_k}^{-1}(b - \lambda)||_p < \infty$ for all $n \ge 1$ and*

$$\kappa_p(T_{n_k}(b - \lambda)) > n_k\, \varphi(n_k)$$

*for infinitely many $n_k \in \mathbf{N}$.*

## III. Symbols with Zeros: Lower Estimates

$$b(t) = \sum_{j=-r}^{s} a_j t^j \text{ - Laurent polynomial, } b(t_0) = 0$$

$$b(z) = \frac{b^{(\alpha)}(t_0)}{\alpha!}(z - t_0)^\alpha + O\left((z - t_0)^{\alpha+1}\right), \quad b^\alpha(t_0) \neq 0$$

$\alpha$ - order of the zero

**Theorem 7** *Let $b$ be a Laurent polynomial and suppose $b$ has a zero of order $\alpha$ at $t_0 \in \mathbf{T}$. Then there is a constant $C \in (0, \infty)$ independent of $n$ such that*

$$\kappa_2(T_n(b)) \geq C\, n^\alpha \text{ for all } n \geq 1.$$

## A class of good test polynomials.

For $j, m \in \mathbf{N}$, consider the Laurent polynomial

$$p_m^j(e^{i\theta}) = \left(1 + e^{i\theta} + \ldots + e^{im\theta}\right)^j . \tag{4}$$

Obviously,

$$p_m^j(e^{it\theta}) = \left(\frac{1 - e^{i(m+1)\theta}}{1 - e^{i\theta}}\right)^j = e^{imj\theta/2} \left(\frac{\sin\frac{m+1}{2}\theta}{\sin\frac{\theta}{2}}\right)^j \tag{5}$$

From (4) we see that $p_m^j \in \mathcal{P}_{mj+1}$. Both (4) and (5) immediately show that

$$||p_m^j||_\infty = (m+1)^j.$$

It's easy to see that

$$||p_m^1||_2^2 = 2\pi(1^2 + \ldots + 1^2) = 2\pi(m+1).$$

**Corollary 2** *Let $b$ be a Laurent polynomial and assume the zeros of $b$ on $\mathbf{T}$ are $t_1, \ldots, t_k$ with the orders $\alpha_1, \ldots, \alpha_k$. Then*

$$\kappa_2(T_n(b)) \geq C\, n^{\max(\alpha_1,\ldots,\alpha_k)} \ \textit{for all } n \geq 1,$$

*where $C \in (0, \infty)$ is a constant independent of $n$.*

## III. Symbols with Zeros: Upper Estimates

For $\beta \in \mathbf{Z}_+$, we define the Laurent polynomials $\xi_\beta$ and $\eta_\beta$ by

$$\xi_\beta(t) = \left(1 - \frac{1}{t}\right)^\beta = \sum_{j=0}^{\beta} (-1)^j \binom{\beta}{j} t^{-j},$$

$$\eta_\beta(t) = (1-t)^\beta = \sum_{j=0}^{\beta} (-1)^j \binom{\beta}{j} t^{j}.$$

**Theorem 8** *Let $\gamma, \delta \in \mathbf{Z}_+$ and let $c$ be a Laurent polynomial without zeros on $\mathbf{T}$ and with winding number zero. Put $b = \xi_\delta \eta_\gamma c$. Then $T_n(b)$ is invertible for all sufficiently large $n$ and there exists a constant $C = C_{\gamma,\delta,c} \in (0,\infty)$ such that*

$$\kappa_2(T_n(b)) \le C\, n^{\gamma+\delta}$$

*for all $n$ large enough.*

## Inside the essential Spectrum

Throughout this section we assume that $b$ is a polynomial. We study the behavior of $||T_n^{-1}(b-\lambda||_2$ in the case where $\lambda \in \mathrm{sp}_{ess}\, T(b) = b(\mathbf{T})$. Clearly, $\lambda \in b(\mathbf{T})$ if and only if $(b-\lambda)$ has zeros on $\mathbf{T}$. We by $S(b)$ denote the points $\lambda$ for which $(b-\lambda)$ has at least two distinct zeros on $\mathbf{T}$. The points in $S(b)$ are met at least twice by $b(t)$ as $t$ traces out the unit circle $\mathbf{T}$. If $\lambda \in b(\mathbb{T}) \setminus S(b)$, then $b(\mathbb{T})$ is an (analytic) arc in a sufficiently small neighborhood of $\lambda$.

**Theorem 9** *Let* $\lambda \in b(\mathbb{T}) \setminus S(b)$ *and*

$$b(t) - \lambda = (t - t_0)^\beta t^k c(t), \quad t \in \mathbf{T},$$

*where* $\beta \in \mathbf{N}$, $k \in \mathbf{Z}$, $c(t) \neq 0$ *for* $t \in \mathbf{T}$, *and* wind $(c, 0) = 0$. Then

$$\kappa_n(T(b-\lambda)) \simeq n^\beta \text{ if } -\beta \leq k \leq 0,$$

and there are constants $C \in (0, \infty)$ and $\alpha \in (0, \infty)$ such that

$$\kappa_n(T(b-\lambda)) \geq C\, e^{\alpha n} \text{ if } k < -\beta \text{ or } k > 0. \qquad (6)$$

**Example 3** *Consider the symbol*

$$b(t) = (t-1)^2 t^k (2.001 + t + 0.49t^{-1}).$$

*Figure 4.3 shows what happens in the five cases* $k = -3, -2, -1, 0, 1$. *In each picture we see the norm* $||T_n^{-1}(b)||_2$ *against* $n$. *We also plotted the shape of the curve* $b(\mathbf{T})$ *in the lower-right corner; the origin is marked by a big dot. As predicted by Theorem 10, the norms increase at least exponentially for* $k = -3$ *and* $k = 1$, *while the growth of the norms is polynomially for* $-2 \leq k \leq 0$. *In the picture in the bottom, we replaced values greater than* $10^{15}$ *by the value* $10^{17}$.

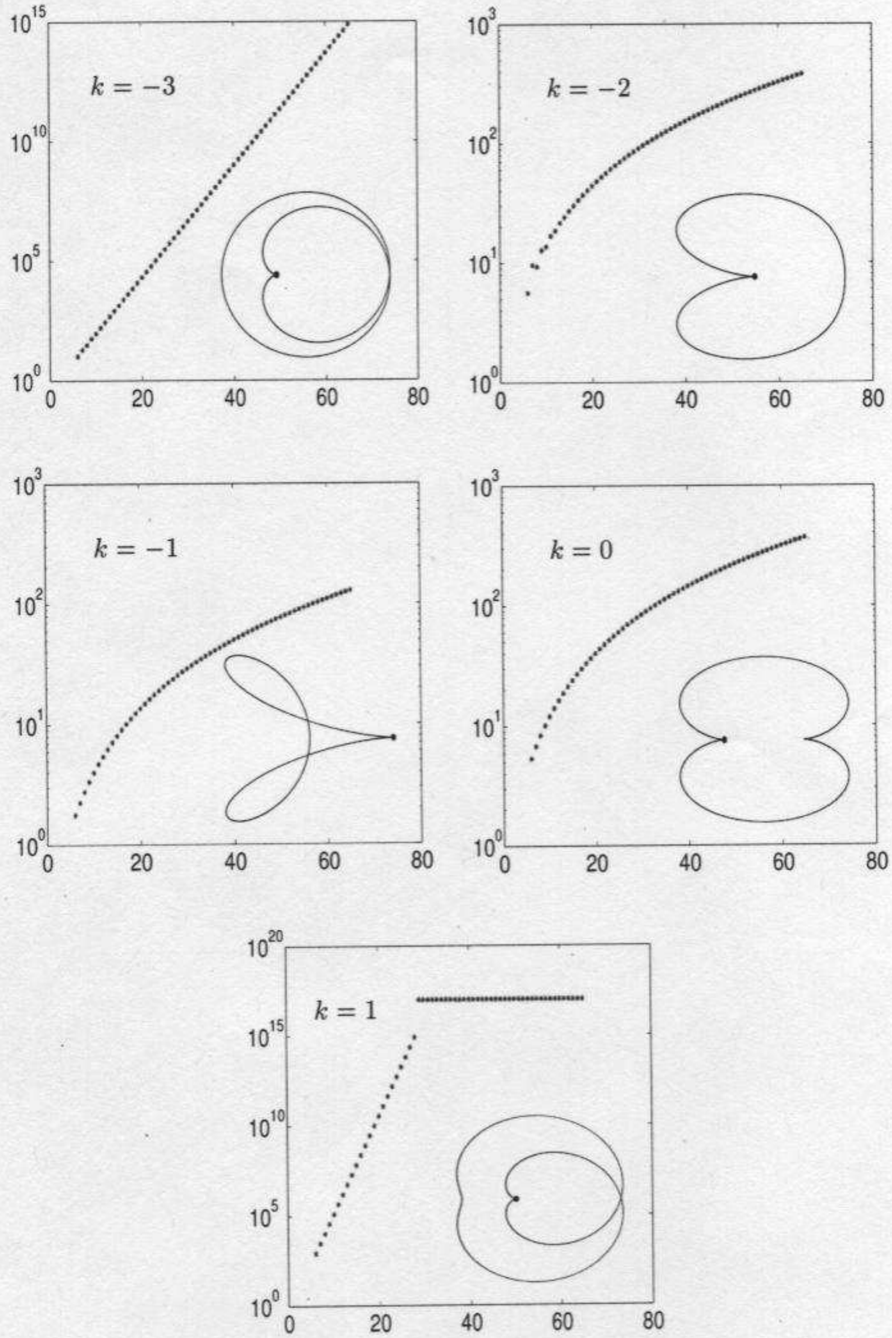


Figure 4.3: Norms $\|T_n^{-1}(b)\|_2$ for several symbols $b$ with zeros.

## Semi Definite Matrices

A matrix $A \in C^{n\times n}$ is said to be *positive semi-definite*

if $\mathbf{Re}\ (Ax, x) \geq 0$ for all $x \in \mathbf{C}^n$ and is called *positive definite* if there is an $\varepsilon > 0$ such that

$\mathbf{Re}\ (Ax, x) \geq \varepsilon ||x||^2$ for all $x \in \mathbf{C^n}$, where $||\cdot||$ is the

$l^2$ norm. If $a \in C(\mathbb{T})$ and $\mathbf{Re}\ a(t) \geq 0$ for all $t \in \mathbf{T}$, then $T_n(a)$ is positive semi-definite, and that if

$a \in C(\mathbb{T})$ and $\mathbf{Re}\ a(t) \geq \varepsilon > 0$ for all $t \in \mathbf{T}$, then $T_n(a)$ is positive definite.

For $a \in C(\mathbf{T})$, let $\mathcal{R}(a) = a(\mathbf{T})$ be the range of $a$, let conv $\mathcal{R}(a)$ stand for the convex hull of $\mathcal{R}(a)$, let conv $\mathcal{R}(a)$ denote the boundary of conv $\mathcal{R}(a)$, and put

$$\text{dist}(0, \text{conv}\mathcal{R}(a)) := \min\{|z| : z \in \text{conv}\mathcal{R}(a)\}.$$

**Proposition 1** *Suppose $a \in C(T)$ does not vanish*

*identically and $\mathcal{R}(a)$ is not a line segment containing*

*the origin in its interior. If*

$$0 \notin \text{conv}\mathcal{R}(a) \quad \textit{or} \quad 0 \in \partial\ \text{conv}\mathcal{R}(a),$$

*then $T_n(a)$ is invertible for all $n \geq 1$.*

**Theorem 10** *Let $a \in C(T)$ and suppose*

$$d := \mathrm{dist}(0, \mathrm{conv}\ \mathrm{a}(\mathrm{t})) > 0.$$

*Then $T(a)$ is invertible on $l^2$ and*

$$||T^{-1}(a)||_2 \leq \frac{1}{d}\left(1 + \sqrt{1 - \frac{d^2}{||a||_\infty}}\right) < \frac{2}{d},$$

*and $T_n(a)$ is invertible for all $n \geq 1$ and*

$$||T_n^{-1}(a)||_2 \leq \frac{1}{d}\left(1 + \sqrt{1 - \frac{d^2}{||a||_\infty}}\right) < \frac{2}{d},$$

## MAIN IDEA:

Let $\mathbf{Re}\, b(t) \geq 0$, and exist function
$g(t) = \sum_{j=-\infty}^{-(n+1)} a_j t^j + \sum_{j=n+1}^{\infty} a_j t^j$, such that
$\mathbf{Re}\, (b(t) + g(t)) > 0$ Then $T_n(b) \equiv T_n(b + g)$ and we can use Theorem 10.

**Theorem 11** *Let $b$ be a Laurent polynomial and suppose $0 \in b(\mathbf{T})$. Assume that $\mathbf{Re}\, b \geq 0$ on $\mathbf{T}$ and that $\mathbf{Re}\, b$ is not identically zero. Then $\mathbf{Re}\, b$ has a finite number of zeros on $\mathbf{T}$ and the orders of these zeros are all even. If $2\alpha$ is the maximal order of the zeros of $\mathbf{Re}\, b$ on $\mathbf{T}$, then*

$$\kappa_n(T_n(b)) \leq D\, n^{2\alpha} \quad \textit{for all } n \geq 1$$

*with some constant $D \in (0, \infty)$ independent of $n$.*

**Theorem 12** *Let $b$ be a real-valued Laurent polynomial and suppose $b$ is not constant. Then $\mathcal{R}(b) = [m, M]$ with $m < M$. If $\lambda \in \{m, M\}$ and the maximal order of the zeros of $b - \lambda$ on $\mathbf{T}$ is $2\alpha$, then*

$$||T_n^{-1}(b - \lambda)||_2 \simeq n^{2\alpha}.$$