Conformal reduction of boundary problems for harmonic functions in a plane domain with strong singularities on the boundary

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We consider the Dirichlet problem for harmonic functions in a bounded plane domain with nonsmooth boundary. The boundary curve belongs to one of the following three classes: sectorial curves, logarithmic spirals and spirals of power type. To study the problem we apply a familiar method of Vekua-Muskhelishvili which consists in using a conformal mapping of the unit disk onto the domain to pull back the problem to a boundary problem for harmonic functions in the disk. This latter reduces in turn to a Toeplitz operator equation on the unit circle with symbol bearing discontinuities of second kind. We develop a constructive invertibility theory for Toeplitz operators and thus derive solvability conditions as well as explicit formulas for solutions.

Elliptic partial differential equations are known to appear in many applied areas of mathematical physics. As but a few we mention

- mechanics of solid medium,
- diffraction theory,
- hydrodynamics,
- gravity theory,
- quantum field theory,

and many others.

In this report we focus on boundary value problems for the Laplace equation in plane domains bounded by nonsmooth curves $\mathcal{C}$. We are primarily interested in domains whose boundaries have a finite number of singular points of oscillating type.

$$z(r) = z_0 + r\ \exp(\imath\varphi(r)) \text{ for } r \in (0, r_0],$$

where

- r is the distance of z and z_0,
- $\varphi(r)$ is a real-valued function which tends to infinity as $r \to 0$ or is bounded while its derivative is unbounded at $r = 0$.

Furthermore, $\varphi(r)$ and $\varphi'(r)$ are allowed to tend to infinity fast enough and our study encompasses domains with nonrectifiable boundary as well.

There is a huge literature devoted to boundary value problems for elliptic equations in domains with nonsmooth boundary.

V.A. Kondratiev.
Boundary value problems for elliptic equations in domains with conical points. Trudy Mosk.Mat.Obshch, 16 (1967), 209–292 pp.

V.A. Kozlov, V.G. Maz'ya, J. Rossman.
Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations. AMS, Providence, R.I, 2000.

V. Maz'ya, S. Nazarov, B. Plamenevskij.
Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains. Vol. 1, 2, Birkhäuser, Basel, 2000.

V. Rabinovich.
Potential type operators on curves with vorticity points. Zeitschrift für Analysis und ihre Anwendungen, 18 (1999), No. 4, 1065-108 pp.

V. Rabinovich, B.W. Schulze and N. Tarkhanov..
Boundary value problems in oscillating cuspidal wedges, Rocky Mountain J. Math, 34, (Fall 2004), No.3, 73 pp.

In most papers one treats piecewise smooth curves with corner points or cusps.

There have been essentially fewer works dealing with more complicated curves $\mathcal{C}$. They mostly focus on qualitative properties, such as

- existence,
- uniqueness and
- stability of solutions with respect to small perturbations.

The present report deals not only with qualitative investigations of boundary value problems in domains whose boundaries strongly oscillate at singular points but also with constructive solution of such problems.

We restrict ourselves to the Dirichlet problem for the Laplace equation

$$\Delta u := (\partial/\partial x)^2 u + (\partial/\partial y)^2 u = 0 \tag{1}$$

in a simply connected domain $\mathcal{D}$ with boundary $\mathcal{C}$ in the plane of variables $(x, y) \in \mathbb{R}^2$. The boundary data are

$$u = u_0 \tag{2}$$

Our approach to the study of elliptic problems in domains with nonsmooth boundary goes back at least as far as

N. Muskhelishvili and I. Vekua.

It consists in reducing the problem in $\mathcal{D}$ to a singular integral equation on the unit circle by means of a conformal mapping of the unit disk onto $\mathcal{D}$. The coefficients of the singular integral equation obtained in this way fail in general to be continuous, for they are intimately connected with the derivative of boundary values of the conformal mapping. This method was successfully used for solving problems in domains with piecewise smooth boundary, where the singular points are corner points or cusps. In this case the coefficients of the mentioned singular integral equation have discontinuities of the first kind. Since the theory of such equations is well elaborated, one has succeeded to construct a sufficiently complete theory of boundary value problems for a number of elliptic equations in domains with piecewise smooth boundary.

Note that by now the theory of singular integral equations (or, in other terms, the theory of Toeplitz operators) with oscillating coefficients is well elaborated, too. This fact gives us some evidence to a reasonable solvability theory for a number of problems in plane domains with oscillating boundary.

The persent report deals with Laplace problems for the Laplace equation for three classes of boundary curves $\mathcal{C}$, namely sectorial curves, logarithmic and power spirals.

Let $\mathcal{D}$ be a simply connected bounded domain in the plane of real variables (x, y). The boundary of $\mathcal{D}$ is a closed Jordan curve which we denote by $\mathcal{C}$. Consider the Dirichlet problem (1), (2) in $\mathcal{D}$ with data u_0 on $\mathcal{C}$. As usual we introduce a complex structure in $\mathbb{R}^2$ by $z = x + \imath y$ and pick a conformal mapping $z = \mathfrak{c}(\zeta)$ of the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ onto the domain $\mathcal{D}$, cf. Riemann mapping theorem. Throught the report

$$\mathfrak{c}'(0) > 0. \tag{3}$$

Then problem (1), (2) can be reformulated as

$$\begin{aligned} \frac{1}{|\mathfrak{c}'(\zeta)|^2} \Delta U &= 0 \quad \text{for} \quad |\zeta| < 1, \\ U &= U_0 \quad \text{for} \quad |\zeta| = 1, \end{aligned} \tag{4}$$

where $U(\zeta) := u(\mathfrak{c}(\zeta))$ and $U_0(\zeta) := u_0(\mathfrak{c}(\zeta))$.

For $1 \leq p < \infty$, we denote by $H^p(\mathbb{D})$ the Hardy space on the unit disk. By the conformal mapping $z = \mathfrak{c}(\zeta)$ the space is transported to the so-called Hardy-Smirnov space $E^p(\mathcal{D})$ of functions on $\mathcal{D}$. A holomorphic function f on $\mathcal{D}$ is said to belong to $E^p(\mathcal{D})$ if

$$\sup_{r\in(0,1)} \int_{\mathcal{C}_r} |f(z)|^p \, |dz| < \infty,$$

where $\mathcal{C}_r$ is the push-forward of the circle $|\zeta| = r$ by $z = \mathfrak{c}(\zeta)$. It is easy to see that $f \in E^p(\mathcal{D})$ if and only if

$$\sqrt[p]{\mathfrak{c}'(\zeta)}\, f(\mathfrak{c}(\zeta)) \in H^p(\mathbb{D}). \tag{5}$$

We will also study boundary value problems in Hardy-Smirnov spaces with weights $E^p(\mathcal{D}, w)$. Set

$$w(\zeta) = \prod_{k=1}^{n} \left(1 - \frac{\zeta}{\zeta_k}\right)^{-\mu_k} \tag{6}$$

for $\zeta \in \mathbb{D}$, where $\{\zeta_1, \ldots, \zeta_n\}$ are pairwise different points on the unit circle. Here, $\mu_1, \ldots, \mu_n$ are real numbers in the interval $(-1/q, 1/p)$, p and q being conjugate exponents, i.e. $1/p + 1/q = 1$. A holomorphic function f in $\mathcal{D}$ is said to lie in $E^p(\mathcal{D}, w)$ if

$$\sup_{r \in (0,1)} \int_{\mathcal{C}_r} |f(z)|^p \, |w(\mathfrak{c}^{-1}(z))|^p \, |dz| < \infty.$$

It is well known that for each harmonic function $u(x, y)$ in $\mathcal{D}$ there is an analytic function $f(z)$ in $\mathcal{D}$ whose real part is u. We therefore look for a solution u of problem (1) and (2), which has the form $u = \Re f$ with $f \in E^p(\mathcal{D}, w)$. The boundary condition $u = u_0$ is understood in the sense of nontangential limit values of u almost everywhere on $\mathcal{C}$.

Definition

Given any Dirichlet data u_0 on $\mathcal{C}$ of class $L^p(\mathcal{C}, w)$ in the sense that

$$\int_{\mathbb{T}} |u_0(\mathfrak{c}(\zeta))|^p \, |w(\zeta)|^p \, |\mathfrak{c}'(\zeta)| \, |d\zeta| < \infty,$$

we shall say that problem (1), (2) possesses a solution in $\Re E^p(\mathcal{D}, w)$ if there is a harmonic function u in $\mathcal{D}$, such that $u = \Re f$ for some $f \in E^p(\mathcal{D}, w)$ and $u = u_0$ on $\mathcal{C}$.

$$f(\mathfrak{c}(\zeta)) = \frac{h^+(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}$$

for $\zeta \in \mathbb{D}$, where h^+ is an analytic function of Hardy class $H^p(\mathbb{D})$.

$$\Re\Big(\frac{h^+(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}\Big) = U_0(\zeta)$$

for $\zeta \in \mathbb{T}$,and

$$U_0(\zeta) = u_0(\mathfrak{c}\,(\zeta))$$

This problem can in turn be reformulated as

$$\frac{1}{2}\Big(\frac{h^+(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}} + \frac{h^-(\zeta)}{\overline{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}}\Big) = U_0(\zeta)$$

for $\zeta \in \mathbb{T}$, where

$$h^-(\zeta) = \overline{h^+(\zeta)} \in \overline{H^p(\mathbb{D})}$$

Finally, we transform the Dirichlet problem to

$$a(\zeta)\,h^{+}(\zeta)+h^{-}(\zeta)=g(\zeta) \tag{7}$$

for $\zeta\in\mathbb{T}$, where

$$a(\zeta)=\frac{\overline{w(\zeta)}}{w(\zeta)}\sqrt[p]{\frac{\overline{\mathfrak{c}'(\zeta)}}{\mathfrak{c}'(\zeta)}}=\frac{\overline{w(\zeta)}}{w(\zeta)}\exp\Big(-\imath\frac{2}{p}\arg\mathfrak{c}'(\zeta)\Big)$$

and $g(\zeta)=2U_0(\zeta)\,\overline{w(\zeta)}\,\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}$.

$$\arg\mathfrak{c}'(\zeta)=\alpha(\mathfrak{c}(\zeta))-\arg\zeta-\frac{\pi}{2}$$

for $\zeta\in\mathbb{T}$, where $\alpha(\mathfrak{c}(\zeta))$ is the angle at which the tangent of $\mathcal{C}$ at the point $z=\mathfrak{c}(\zeta)$ intersects the real axis. Note that $g\in L^p(\mathbb{T})$.

Remind that

$$h^{+}(\zeta)\in H^p(\mathbb{D}),\quad h^{-}(\zeta)\in\overline{H^p(\mathbb{D})}$$

Let now

$$(S_{\mathbb{T}}g)(\zeta) := \frac{1}{\pi \imath} \int_{\mathbb{T}} \frac{g(\zeta')}{\zeta' - \zeta} \, d\zeta', \quad \zeta \in \mathbb{T},$$

stand for the singular Cauchy integral. If $1 < p < \infty$, then $S_{\mathbb{T}}$ is a bounded operator in $L^p(\mathbb{T})$, and the operators

$$P_{\mathbb{T}}^{\pm} := \frac{1}{2} (I \pm S_{\mathbb{T}})$$

prove to be continuous projections in $L^p(\mathbb{T})$ called analytic projections.

$$P_{\mathbb{T}}^{\pm} \, L^p(\mathbb{T}) = H^{p\pm},$$

whence $P_{\mathbb{T}}^{\pm} \, H^{p\pm} = H^{p\pm}$ and $P_{\mathbb{T}}^{\pm} \, H^{p\mp} = 0$.

$$H^{p^+} = H^p(\mathbb{D}), \quad H^{p-} \oplus \{c\} = \overline{H^p(\mathbb{D})}$$

On applying $P_{\mathbb{T}}^{+}$ to both sides of equality (7) and taking into account that $(P_{\mathbb{T}}^{+} h^{-})(\zeta) = h^{-}(\infty)$ and $h^{-}(\infty) = \overline{h^{+}(0)} = h^{+}(0)$, we get

$$(T(a)h^{+})(\zeta) + h^{+}(0) = g^{+}(\zeta) \qquad (8)$$

for $\zeta \in \mathbb{T}$, where

$$T(a) := P_{\mathbb{T}}^{+} a P_{\mathbb{T}}^{+}$$

is a Toeplitz operator with symbol a on $L^{p}(\mathbb{T})$ and $g^{+}(\zeta) = (P_{\mathbb{T}}^{+} g)(\zeta)$.

Theorem

- *If $u = \Re f$ with $f \in E^{p}(\mathcal{D}, w)$ is a solution of the Dirichlet problem in $\mathcal{D}$, then $h^{+}(\zeta) = w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}\, f(\mathfrak{c}(\zeta))$ is a solution of system (8)*
- *If $h^{+} \in H^{p+}$ is a solution of (8), then $u(\mathfrak{c}(\zeta)) = \Re\left(h^{+}(\zeta)/w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}\right)$ is a solution of the Dirichlet problem in $\mathcal{D}$.*

Corollary (*Exat formula*)

If the operator $T(a)$ is invertible on the space H^{p+} and

$$\left(T(a)^{-1}1\right)(0) = 1, \tag{9}$$

then the Dirichlet problem in $\mathcal{D}$ has a unique solution of the form

$$u(z) = \Re\Big(\frac{1}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}\Big((T(a)^{-1}g^+)(\zeta) - \frac{1}{2}(T(a)^{-1}g^+)(0)\,(T(a)^{-1}1)(\zeta)\Big)\Big),$$

with $z = \mathfrak{c}(\zeta)$, where $g^+ = P^+_{\mathbb{T}}\left(2u_0(\mathfrak{c}(\zeta))\,\overline{w(\zeta)}\,\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}\right)$.

Remark

Condition (9) is actually fulfilled in all cases to be treated in this work.

Proof of Corollary *Exat formula*

Applying the operator $T(a)^{-1}$ to (8) yields

$$h^{+}(\zeta) + T(a)^{-1}h^{+}(0) = \left(T(a)^{-1}g^{+}\right)(\zeta)$$

for all $\zeta \in \mathbb{T}$. Since both sides of the equality extend to holomorphic functions in the disk, we can set $\zeta = 0$, obtaining

$$2\,h^{+}(0) = \left(T(a)^{-1}g^{+}\right)(0)$$

by (9). Hence it follows that the solution of (8) is unique and it has the form

$$h^{+}(\zeta) = \left(T(a)^{-1}g^{+}\right)(\zeta) - \frac{1}{2}\left(T(a)^{-1}g^{+}\right)(0)(T(a)^{-1}1)(\zeta),$$

as desired.

Factorization of symbols

A bounded linear operator A on a Hilbert space H is said to be normally solvable if its range $\mathrm{im}\, A$ is closed. A normally solvable operator is called Fredholm if its kernel and cokernel are finite dimensional. In this case the index of A is introduced as

$$\mathrm{ind}\, A := \alpha(A) - \beta(A),$$

where $\alpha(A) = \dim \ker A$ and $\beta(A) = \dim \mathrm{coker}\, A$.

The symbol $a(\zeta)$ of a Toeplitz operator $T(a)$ is said to admit a p-factorisation, with $1 < p < \infty$, if it can be represented in the form

$$a(\zeta) = a^+(\zeta)\zeta^\kappa a^-(\zeta), \tag{10}$$

where κ is an integer number,

$$\begin{array}{rclrcl} a^+ & \in & H^{q+}, & a^- & \in & H^{p-} \oplus \{c\}, \\ 1/a^+ & \in & H^{p+}, & 1/a^- & \in & H^{q-} \oplus \{c\}, \end{array} \tag{11}$$

p and q are conjugate exponents (i.e. $1/p + 1/q = 1$), and

$$(1/a^+)S_{\mathbb{T}}(1/a^-) \text{ is a bounded operator on } L^p(\mathbb{T}). \tag{12}$$

The functions a^+ and a^- in (10) are determined uniquely up to a constant factor. On putting the additional condition $a^-(\infty) = 1$ one determines the factorisation uniquely.

Theorem (*Simonenko*)

An operator $T(a)$ is Fredholm in the space H^{p+} if and only if the symbol $a(\zeta)$ admits a p-factorisation. If $T(a)$ is Fredholm, then $\operatorname{ind} T(a) = -\kappa$.

Corollary (*Invertibility*)

An operator $T(a)$ is invertible on H^{p+} if and only if the symbol $a(\zeta)$ admits a p-factorisation with $\kappa = 0$. In this case

$$(T(a))^{-1} = (1/a^{+})P_{\mathbb{T}}^{+}(1/a^{-}).$$

Given a nonvanishing function $a \in C(\mathbb{T})$, we denote by $\mathrm{ind}_{a(\mathbb{T})}(0)$ the winding number of the curve $a(\mathbb{T})$ about the origin, or the index of the origin with respect to $a(\mathbb{T})$.

Theorem

Suppose $a \in C(\mathbb{T})$. Then the operator $T(a)$ is Fredholm on the space H^{p+} if and only if $a(\zeta) \neq 0$ for all $\zeta \in \mathbb{T}$. Under this condition the index of $T(a)$ is given by

$$\mathrm{ind}\, T(a) = -\mathrm{ind}_{a(\mathbb{T})}(0).$$

We now introduce the concept of sectoriality which is of crucial importance in this report.

Definition

A function $a \in L^\infty(\mathbb{T})$ is called p-sectorial if $\operatorname{ess\,inf} |a(\zeta)| > 0$ and there is a real number φ_0 such that

$$\sup_{\zeta \in \mathbb{T}} |\arg(\exp(\imath\varphi_0)a(\zeta))| < \frac{\pi}{\max\{p, q\}} \tag{13}$$

for all $\zeta \in \mathbb{T}$.

Definition

A function $a \in L^\infty(\mathbb{T})$ is said to be locally p-sectorial if $\operatorname{ess\,inf} |a(\zeta)| > 0$ and for any $\zeta_0 \in \mathbb{T}$ there is an open arc containing ζ_0, such that (13) is satisfied for all ζ in the arc with some $\varphi_0 \in \mathbb{R}$ depending on ζ_0.

Theorem

- *If* $a(\zeta)$ *is a* p*-sectorial symbol, then the operator* $T(a)$ *is invertible in the space* H^{p+}*.*
- *If* $a(\zeta)$ *is a locally* p*-sectorial symbol, then* $T(a)$ *is a Fredholm operator in* H^{p+}*.*

Suppose $h \in GH^{\infty +}$, that is $h \in H^{\infty +}$ and $1/h \in H^{\infty +}$. Then the operator $T(h)$ is invertible in H^{p+}, $(T(h))^{-1} = T(1/h)$. Analogously, if $h \in GH^{\infty -}$ then the operator $T(h)$ is invertible in H^{p+} and $(T(h))^{-1} = T(1/h)$.

Theorem

- *Let $a(\zeta) = h(\zeta)a_0(\zeta)$, where $h \in GH^{\infty \pm}$ and $a_0 \in L^\infty(\mathbb{T})$. Then the operator $T(a)$ is Fredholm in the space H^{p+} if and only if the operator $T(a_0)$ is Fredholm, in which case* $\operatorname{ind} T(a) = \operatorname{ind} T(a_0)$.
- *Let $a(\zeta) = c(\zeta)a_0(\zeta)$, where $c \in C(\mathbb{T})$ and $a_0 \in L^\infty(\mathbb{T})$. Then $T(a)$ is Fredholm in H^{p+} if and only if $c(\zeta)$ vanishes at no point of $\mathbb{T}$ and $T(a_0)$ is Fredholm, in which case*

$$\operatorname{ind} T(a) = \operatorname{ind} T(a_0) - \operatorname{ind}_{c(\mathbb{T})}(0).$$

We give a brief summary on Toeplitz operators with symbols having discontinuities of the first kind. Let PC stand for the space of all piecewise continuous functions on $\mathbb{T}$ which have at most finitely many jumps. Suppose $a(\zeta) \in PC$ and $\zeta_1, \ldots, \zeta_n$ are the points of discontinuity of a. Given any $\mathfrak{f} \in \mathbb{C}$ and $\zeta_0 \in \mathbb{T}$, we introduce a function $a_{\mathfrak{f},\zeta_0} \in PC$ by

$$a_{\mathfrak{f},\zeta_0}(\zeta) := \exp\Big(\imath \mathfrak{f} \arg\Big(-\frac{\zeta}{\zeta_0}\Big)\Big)$$

for $\zeta \in \mathbb{T}$, where $\arg z \in (-\pi, \pi]$. It is easily seen that $a_{\mathfrak{f},\zeta_0}$ has at most one point of discontinuity at $\zeta = \zeta_0$, with jump

$$\begin{aligned} a_{\mathfrak{f},\zeta_0}(e^{\imath 0-}\zeta_0) &= \exp(\imath \mathfrak{f} \pi), \\ a_{\mathfrak{f},\zeta_0}(e^{\imath 0+}\zeta_0) &= \exp(-\imath \mathfrak{f} \pi). \end{aligned}$$

If $a(e^{\imath 0\pm}\zeta_k) \neq 0$ for all $k = 1, \ldots, n$, then there are complex numbers $\mathfrak{f}_k$ with the property that

$$\frac{a(e^{\imath 0-}\zeta_k)}{a(e^{\imath 0+}\zeta_k)} = \exp\left(2\imath \mathfrak{f}_k \pi\right),$$

and so

$$a(\zeta) = c(\zeta) \prod_{k=1}^{n} a_{\mathfrak{f}_k,\zeta_k}(\zeta), \tag{14}$$

where $c \in C(\mathbb{T})$.

Theorem (PC)

Let $a(\zeta) \in PC$. Then the operator $T(a)$ is Fredholm in H^{p+} if and only if

- $a(e^{\imath 0\pm}\zeta) \neq 0$ *for all* $\zeta \in \mathbb{T}$;
- *there are integer numbers* κ_k *such that* $\kappa_k - \frac{1}{q} < \Re \mathfrak{f}_k < \kappa_k + \frac{1}{p}$.

Under conditions this Theorem, the index of the operator $T(a)$ in H^{p+} is actually given by

$$\mathrm{ind}\, T(a) = -\Big(\mathrm{ind}_{c(\mathbb{T})}(0) + \sum_{k=1}^{n} \kappa_k\Big), \tag{15}$$

Sectorial curves

In this section we consider a simply connected domain $\mathcal{D} \Subset \mathbb{R}^2$ whose boundary $\mathcal{C}$ is smooth away from a finite number of points. By this is meant that $\mathcal{C}$ is a Jordan curve of the form

$$\mathcal{C} = \bigcup_{k=1}^{n} \mathcal{C}_k,$$

where $\mathcal{C}_k = [z_{k-1}, z_k]$ is an arc with initial point z_{k-1} and endpoint z_k which are located after each other in positive direction on $\mathcal{C}$, and $z_n = z_0$. Moreover, (z_{k-1}, z_k) is smooth for all k.

Definition

The curve $\mathcal{C}$ is called p-sectorial if, for each $k = 1, \ldots, n$, there is a neighbourhood (z_k^-, z_k^+) of z_k on $\mathcal{C}$ and a real number φ_k, such that

$$\sup_{z \in (z_k^-, z_k^+) \setminus \{z_k\}} |\alpha(z) - \varphi_k| < \begin{cases} \frac{\pi}{2}, & \text{if} \quad p \geq 2, \\ \frac{\pi}{2}(p-1), & \text{if} \quad 1 < p < 2, \end{cases} \tag{16}$$

where $\alpha(z)$ is the angle at which the tangent of $\mathcal{C}$ at the point z intersects the real axis.

Example (1)

If z_k is a corner point of $\mathcal{C}$, then the angle at which the tangent of $\mathcal{C}$ at z intersects the real axis has jump $j_k < \pi$ when z passes through z_k. It follows that (16) is fulfilled at z_k with a suitable φ_k, if $p \geq 2$, and is fulfilled if moreover $j_k < (p-1)\pi$, if $1 < p < 2$.

If z_k is a cuspidal point of $\mathcal{C}$, then the angle has jump $j_k = \pi$ when z passes through z_k. Hence, condition (16) is violated, i.e. cuspidal points are prohibited for sectorial curves.

Example (2)

$$z(e^{i\theta}) = \theta + i\,\delta|\theta|^{\alpha+1}\sin|\theta|^{-\alpha}, \quad |\theta| < \epsilon, \quad \alpha > 0, \quad \delta \in R.$$

If $p \geq 2$, this curve is $p-$ sectorial for any α.

If $1 < p < 2$ and $|\delta\alpha| < \tan\frac{\pi}{2}(p-1)$, then this curve is $p-$ sectorial.

Example (3)

$$z(e^{i\theta}) = \theta(1 + \imath\,\sin|\theta|^{-\alpha}), \quad |\theta| < \epsilon, \quad \alpha > 0.$$

This curve is not $p-$ sectorial.

Dirichlet data on sectorial curves

Theorem

Suppose $\mathcal{C}$ is p-sectorial for $1 < p < \infty$ and $w(\zeta) \equiv 1$. Then the Toeplitz operator (8) corresponding to this curve is invertible.

We shall have to proof that the Toeplitz operator with symbol

$$a(\zeta) = \exp\Big(- \imath \frac{2}{p} \arg \mathfrak{c}'(\zeta) \Big)$$

is invertible. Recall that $\arg \mathfrak{c}'(\zeta) = \alpha(\mathfrak{c}(\zeta)) - \arg \zeta - \frac{\pi}{2}$ for $\zeta \in \mathbb{T}$. The idea of the proof is to represent the symbol in the form

$$a(\zeta) = c(\zeta) a_0(\zeta),$$

where a_0 is p-sectorial and $c \in C(\mathbb{T})$ is such that $\mathrm{ind}_{c(\mathbb{T})}(0) = 0$.

Theorem (*p*-sectorial)

Let $\mathcal{C}$ be a p-sectorial curve. Then the Dirichlet problem has a unique solution $u = \Re f$ with $f \in E^p(\mathcal{D})$ and this solution has a following form in $\mathcal{D}$

$$u(z) = \Re \int_{\mathbb{T}} \frac{1}{2\pi\imath} \frac{\zeta + \mathfrak{c}^{-1}(z)}{\zeta - \mathfrak{c}^{-1}(z)} u_0(\mathfrak{c}(\zeta)) \frac{d\zeta}{\zeta}$$

for $z \in \mathcal{D}$.

Begin proof of Theorem *p-sectorial*

According to previous Theorem, $a(\zeta)$ admits of p-factorisation of the kind $a(\zeta) = a^+(\zeta)a^-(\zeta)$. On other hand

$$a(\zeta) = \left(\frac{\overline{\mathfrak{c}'(\zeta)}}{\mathfrak{c}'(\zeta)}\right)^{1/p}$$

for $\zeta \in \mathbb{T}$. In the case of p-sectorial curves the angle $\alpha(z)$ is bounded, and so the curve $\mathcal{C}$ is rectifiable. By a well-known result the derivative $\mathfrak{c}'(\zeta)$ belongs to H^{1+}, whence $\sqrt[p]{\mathfrak{c}'(\zeta)} \in H^{p+}$ and $\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}} \in H^{p-} \oplus \{c\}$. Comparing this with $a(\zeta) = a^+(\zeta)a^-(\zeta)$ we get

$$\sqrt[p]{\mathfrak{c}'(\zeta)}a^+(\zeta) = \overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}(1/a^-(\zeta)).$$

End proof of Theorem *p-sectorial*

The left-hand side of this equality belongs to H^{1+} and the right-hand side to $H^{1-} \oplus \{c\}$. Hence it follows that

$$\begin{aligned} \sqrt[p]{\mathfrak{c}'(\zeta)}\, a^{+}(\zeta) &= c, \\ \overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}\,(1/a^{-}(\zeta)) &= c \end{aligned}$$

where c is a complex constant. The factorisation $a(\zeta) = a^{+}(\zeta)a^{-}(\zeta)$ with

$$\begin{aligned} a^{+}(\zeta) &= c\left(1/\sqrt[p]{\mathfrak{c}'(\zeta)}\right), \\ a^{-}(\zeta)) &= \frac{1}{c}\,\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}} \end{aligned}$$

and

$$(T(a))^{-1} = \sqrt[p]{\mathfrak{c}'(\zeta)}\, P^{+}_{\mathbb{T}}(1/\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}),$$

which is due to Corollary *Invertibility*.

Dirichlet data on logarithmic spirals

Let Σ_a be a horizontal half-strip of the form
$\Sigma_a = \{\mathfrak{z} \in \mathbb{C} : \Re\mathfrak{z} > 0,\ \Im\mathfrak{z} \in (0, a)\},\ a > 0.$

$$z = \ell_\varphi(\mathfrak{z}) := \exp(-e^{\imath\varphi}\mathfrak{z})$$

of the half-strip into the complex plane $\mathbb{C}_z$.

The mapping $z = \ell_\varphi(\mathfrak{z})$ is conformal if and only if $a < 2\pi \cos\varphi$.

For $\upsilon \in [0, a]$, set $\mathcal{R}_\upsilon = \{\mathfrak{z} \in \mathbb{C} : \Re\mathfrak{z} \geq 0,\ \Im\mathfrak{z} = \upsilon\}$. Then the curve $\mathcal{S}_{\upsilon,\varphi} := \ell_\varphi(\mathcal{R}_\upsilon)$ is a spiral.

$$r = \left(\exp\frac{\upsilon}{\sin\varphi}\right)\exp\left(\vartheta\ \cot\varphi\right), \tag{17}$$

$$\mathcal{C}_{a,\varphi} := \mathcal{S}_{0,\varphi} \cup \mathcal{S}_{a,\varphi} \cup b_{a,\varphi},$$

where $\mathcal{S}_{0,\varphi}$ and $\mathcal{S}_{a,\varphi}$ are given by (17) and the arc $b_{a,\varphi}$ by
$z = \exp\left(-e^{\imath\varphi}\imath\upsilon\right)$ with $\upsilon \in [0, a]$.

The conformal mapping $z = \mathfrak{c}_{a,\varphi}(\zeta)$ of $\mathbb{D}$ onto $\mathcal{D}_{a,\varphi}$ given by

$$\mathfrak{c}_{a,\varphi}(\zeta) = \exp\Big(- e^{\imath\varphi} \frac{a}{\pi} \ln \Big(\eta + \sqrt{\eta^2 - 1}\Big) \Big) \tag{18}$$

with $\eta = \eta(\zeta)$, $\eta(\zeta) = \frac{1}{i} \; \frac{\zeta + 1}{\zeta - 1}$.

$$\arg \mathfrak{c}'_{a,\varphi}(\zeta) = \begin{cases} \varphi - \frac{\pi}{2} - \arg\zeta - a\cos\varphi - \frac{a}{\pi}\sin\varphi \, \ln|\eta + \sqrt{\eta^2 - 1}|, \\ \qquad\qquad \textbf{if} \quad \zeta \in (1, \imath), \\ \varphi - \arg\zeta - \frac{a}{\pi}\cos\varphi \; \arccos\eta, \\ \qquad\qquad \textbf{if} \quad \zeta \in [\imath, -\imath], \\ \varphi + \frac{\pi}{2} - \arg\zeta - \frac{a}{\pi}\sin\varphi \, \ln(\eta + \sqrt{\eta^2 - 1}), \\ \qquad\qquad \textbf{if} \quad \zeta \in (-\imath, 1). \end{cases} \tag{19}$$

We introduce the Toeplitz operator $T(a_{a,\varphi})$ with symbol

$$a_{a,\varphi}(\zeta) = \exp\Big(- \imath \frac{2}{p} \arg \mathfrak{c}'_{a,\varphi}(\zeta)\Big),$$

Theorem (*Logarithmic spiral*)

The operator $T(a_{a,\varphi})$ *in* H^{p+} *with* $1 < p < \infty$ *is Fredholm if and only if*

$$p \neq \frac{a}{\pi} \cos \varphi.$$

- *If* $p > \dfrac{a}{\pi} \cos \varphi$, *then* $T(a_{a,\varphi})$ *is invertible in the space* H^{p+}.
- *If* $p < \dfrac{a}{\pi} \cos \varphi$, *then* $\operatorname{ind} T(a_{a,\varphi}) = 1$.

Begin proof of Theorem *Logarithmic spiral*

Introduce the function

$$h_{a,\varphi}(\zeta) := \exp\Big(-\imath\frac{2}{p}\Big(\varphi - \frac{a}{\pi}\sin\varphi\,\ln(\eta + \sqrt{\eta^2-1})\Big)\Big),$$

which obviously belongs to $GH^{\infty+}$. Indeed,

$$|h_{a,\varphi}(\zeta)| = \begin{cases} \exp\Big(-\frac{2}{p}\,a\,\sin\varphi\Big), & \text{if} \quad \zeta \in (1,\imath), \\ 1, & \text{if} \quad \zeta \in (-\imath, 1), \end{cases}$$

$$a_0(\zeta) = \frac{a_{a,\varphi}(\zeta)}{h_{a,\varphi}(\zeta)}$$

End proof of Theorem *Logarithmic spiral*

Hence $a_0(\zeta)$ is a PC function with discontinuity points $\{1, \imath, -\imath\}$. One verifies readily that

$$\begin{aligned}
&\arg a_0(e^{\imath 0-}\,1) = \tfrac{2}{p}\,\tfrac{3}{2}\pi, && \arg a_0(e^{\imath 0+}\,1) = \tfrac{2}{p}\left(\tfrac{\pi}{2} + a\cos\varphi\right),\\
&\arg a_0(e^{\imath 0-}\,\imath) = \tfrac{2}{p}\,(\pi + a\cos\varphi), && \arg a_0(e^{\imath 0+}\,\imath) = \tfrac{2}{p}\left(\tfrac{\pi}{2} + a\cos\varphi\right),\\
&\arg a_0(e^{\imath 0+}(-\imath)) = \tfrac{2}{p}\,\tfrac{3}{2}\pi, && \arg a_0(e^{\imath 0+}(-\imath)) = \tfrac{2}{p}\pi.
\end{aligned}$$

We thus conclude that $a_0(\zeta)$ possesses a representation (14) with $\zeta_1 = 1$, $\zeta_2 = \imath$, $\zeta_3 = -\imath$ and

$$\mathfrak{f}_1 = \frac{1}{p}\left(1 - \frac{a}{\pi}\cos\varphi\right), \qquad \mathfrak{f}_2 = \mathfrak{f}_3 = \frac{1}{2}\,\frac{1}{p}.$$

Observe that

$$-\frac{1}{q} < \mathfrak{f}_2 = \mathfrak{f}_3 < \frac{1}{p}.$$

Theorem

- If $p > \frac{a}{\pi} \cos\varphi$, then the Dirichlet problem has a unique solution in $\Re E^p(\mathcal{D}_{a,\varphi})$ given by

$$u(z) = \Re \int_{\mathbb{T}} \frac{1}{2\pi\imath} \frac{\zeta + \mathfrak{c}_{a,\varphi}^{-1}(z)}{\zeta - \mathfrak{c}_{a,\varphi}^{-1}(z)} \, u_0(\mathfrak{c}_{a,\varphi}(\zeta)) \, \frac{d\zeta}{\zeta}$$

  for $z \in \mathcal{D}_{a,\varphi}$.

- If $p < \frac{a}{\pi} \cos\varphi$, then the Dirichlet problem has infinitely many solutions in $\Re E^p(\mathcal{D}_{a,\varphi})$ given by

$$u(z) = \Re\Big(c \, \frac{1 + \mathfrak{c}_{a,\varphi}^{-1}(z)}{1 - \mathfrak{c}_{a,\varphi}^{-1}(z)} - \frac{\mathfrak{c}_{a,\varphi}^{-1}(z)}{1 - \mathfrak{c}_{a,\varphi}^{-1}(z)} \int_{\mathbb{T}} \frac{1}{\pi\imath} \frac{\zeta - 1}{\zeta - \mathfrak{c}_{a,\varphi}^{-1}(z)} \, u_0(\mathfrak{c}_{a,\varphi}(\zeta)) \, \frac{d\zeta}{\zeta} \Big)$$

  for $z \in \mathcal{D}_{a,\varphi}$, where c is an arbitrary real constant.

Remark

If $p = (a/\pi)\cos\varphi$ *then the operator* $T(a_{a,\varphi})$ *is not Fredholm. One can show that it has zero null space and dense range in this case. What is still lacking is an explicit description of the range.*

Dirichlet data on spirals of power type

Let $a > 0$ and $\gamma > 0$. Consider the domain $\mathcal{D}_{a,\gamma}$ in the plane of complex variable θ that is bounded by curves

$$\begin{aligned}\mathcal{S}_{0,\gamma} &= \{\theta \in \mathbb{C} : \Re\theta \geq 1, \ \Im\theta = 0\}, \\ \mathcal{S}_{a,\gamma} &= \{\theta \in \mathbb{C} : \theta = (\Re\theta + \imath a)^{\gamma}, \ \Re\theta \geq 1\}, \\ b_{a,\gamma} &= \{\theta \in \mathbb{C} : \theta = (1 + \imath\Im\theta)^{\gamma}, \ \Im\theta \in (0, a)\}.\end{aligned}$$

The boundary of $\mathcal{D}_{a,\gamma}$ is thus the composite curve $\mathcal{C}_{a,\gamma} = \mathcal{S}_{0,\gamma} \cup \mathcal{S}_{a,\gamma} \cup b_{a,\gamma}$.
Given any $\delta > 0$, we define

$$z = h_{\delta}(\theta) := \frac{\exp(\imath\theta)}{\theta^{\delta}}$$

for $\theta \in \mathcal{D}_{a,\gamma}$. This function maps $\mathcal{D}_{a,\gamma}$ onto a domain $\mathcal{D}_{a,\gamma,\delta}$.

Consider the curve $h_\delta(\mathcal{S}_{0,\gamma})$ in the z-plane. Introduce the polar coordinates $r = |z|$ and $\vartheta = \arg z$.

$$\begin{cases} r &= (\Re\theta)^{-\delta}, \\ \vartheta &= \Re\theta, \end{cases}$$

and so the equation of $h_\delta(\mathcal{S}_{0,\gamma})$ reduces to

$$r = \vartheta^{-\delta}$$

with $\vartheta \geq 1$. In this way we obtain what will be referred to as the power spiral.

Note that the curve $h_\delta(\mathcal{S}_{0,\gamma})$ is rectifiable if and only if $\delta > 1$. Indeed, the integral

$$\begin{aligned} L(h_\delta(\mathcal{S}_{0,\gamma})) &= \int_1^\infty \sqrt{(r(\vartheta))^2 + (r'(\vartheta))^2}\, d\vartheta \\ &= \int_1^\infty \vartheta^{-\delta} \sqrt{1 + \frac{\delta^2}{\vartheta^2}}\, d\vartheta \end{aligned}$$

is finite if and only if $\delta > 1$.

Theorem

Assume $\gamma \in (0, 1/2]$ *and* $\delta > 0$. *If* $a > 0$ *is small enough, then* $h_\delta : \mathcal{D}_{a,\gamma} \to \mathcal{D}_{a,\gamma,\delta}$ *is a conformal mapping.*

The conformal mapping $z = \mathfrak{c}_{a,\gamma,\delta}(\zeta)$ of $\mathbb{D}$ onto $\mathcal{D}_{a,\gamma,\delta}$ given by

$$\mathfrak{c}_{a,\gamma,\delta}(\zeta) = \frac{\exp \imath \Big(\frac{a}{\pi} \ln \Big(\eta + \sqrt{\eta^2 - 1} \Big) + 1 \Big)^{\gamma}}{\Big(\frac{a}{\pi} \ln \Big(\eta + \sqrt{\eta^2 - 1} \Big) + 1 \Big)^{\gamma\delta}} \tag{20}$$

with $\eta = \eta(\zeta), \;\; 2(\zeta) = \frac{1}{i} \; \frac{\zeta + 1}{\zeta - 1}$.

Introduce the function $\mathfrak{z}(\zeta) = \frac{a}{\pi} \ln \Big(\eta + \sqrt{\eta^2 - 1} \Big) + 1$ of $\zeta \in \mathbb{D}$.

$$\mathfrak{c}'_{a,\gamma,\delta}(\zeta) = \sqrt{2}\imath\gamma \, \frac{a}{\pi} \left(1 + \imath\delta \left(\mathfrak{z}(\zeta)\right)^{-\gamma}\right) \frac{\left(\mathfrak{z}(\zeta)\right)^{\gamma - \gamma\delta - 1}}{\sqrt{1 + \zeta^2}\,(1 - \zeta)} \exp \imath(\mathfrak{z}(\zeta))^{\gamma}. \tag{21}$$

We consider the problem in space $\Re E_p(\mathcal{D}_{a,\gamma,\delta}, w)$, where $w = (1-\zeta)^{-\mu}$

Theorem (*Power spiral*)

Let $\gamma \in (0, 1/2]$, $\delta > 0$ *and let* $\mathfrak{c}_{a,\gamma,\delta}(\zeta)$ *be the conformal mapping of* $\mathbb{D}$ *onto* $\mathcal{D}_{a,\gamma,\delta}$ *given by (20). Then operator* $T(a_{a,\gamma,\delta})$ *corresponding to region* $\mathcal{D}_{a,\gamma,\delta}$ *is Fredholm if and only if*

$$\mu \neq 0$$

.

- *If* $-\frac{1}{q} < \mu < 0$, *then the operator* $T(a_{a,\gamma,\delta})$ *is invertible.*
- *If* $0 < \mu < \frac{1}{p}$, *then operator* $T(a_{a,\gamma,\delta})$ *is Fredholm and* $\operatorname{ind} T(a_{a,\gamma,\delta}) = -1$.

Begin proof of Theorem *Power spiral*

Consider the Toeplitz operator $T(a_{a,\gamma,\delta})$ with symbol

$$\begin{aligned} a_{a,\gamma,\delta}(\zeta) &= \exp\Big(-\imath\frac{2}{p}\arg \mathfrak{c}'_{a,\gamma,\delta}(\zeta)\Big)\frac{\overline{w(\zeta)}}{w(\zeta)} \\ &= \exp\Big(-\imath\frac{2}{p}\arg \mathfrak{c}'_{a,\gamma,\delta}(\zeta)\Big)\frac{(1-1/\zeta)^{-\mu}}{(1-\zeta)^{-\mu}} \\ &= \exp\Big(-\imath\frac{2}{p}\arg \mathfrak{c}'_{a,\gamma,\delta}(\zeta)\Big)(-\zeta)^{\mu} \end{aligned}$$

Set

$$h_{a,\gamma,\delta}(\zeta) = \exp\Big(-\imath\frac{2}{p}(\mathfrak{z}(\zeta))^{\gamma}\Big) \in GH^{\infty+}$$

$$a_0(\zeta) = \frac{a_{a,\gamma,\delta}(\zeta)}{h_{a,\gamma,\delta}(\zeta)}$$

End proof of Theorem *Power spiral*

$$\begin{aligned}
\lim_{\zeta\to e^{\imath 0-}1} \arg a_0(\zeta) - \lim_{\zeta\to e^{\imath 0+}1} \arg a_0(\zeta) &= 2\pi\left(\frac{1}{p}+\mu\right),\\
\lim_{\zeta\to e^{\imath 0-}\imath} \arg a_0(\zeta) - \lim_{\zeta\to e^{\imath 0+}\imath} \arg a_0(\zeta) &= \frac{\pi}{p},\\
\lim_{\zeta\to e^{\imath 0-}(-\imath)} \arg a_0(\zeta) - \lim_{\zeta\to e^{\imath 0+}(-\imath)} \arg a_0(\zeta) &= \frac{\pi}{p}.
\end{aligned}$$

$\zeta_1 = 1$, $\zeta_2 = \imath$, $\zeta_3 = -\imath$, and

$$\mathfrak{f}_1 = \frac{1}{p} + \mu, \qquad \mathfrak{f}_2 = \mathfrak{f}_3 = \frac{1}{2}\,\frac{1}{p},$$

Thus according to Theorem *PC* operator $T(a_{a,\gamma,\delta})$ is Fredholm if and only if $\mu \neq 0$.

Theorem (*Solution*)

Let $\gamma \in (0, 1/2]$, $\delta > 0$ *and let* $z = \mathfrak{c}_{a,\gamma,\delta}(\zeta)$ *be the conformal mapping of* $\mathbb{D}$ *onto* $\mathcal{D}_{a,\gamma,\delta}$ *given by (20).*

- *If* $-\dfrac{1}{q} < \mu < 0$, *then the Dirichlet problem has a unique solution in* $\Re E^p(\mathcal{D}_{a,\gamma,\delta}, w)$ *given by*

$$u(z) = \Re \int_{\mathbb{T}} \frac{1}{2\pi\imath} \frac{\zeta + \mathfrak{c}^{-1}_{a,\gamma,\delta}(z)}{\zeta - \mathfrak{c}^{-1}_{a,\gamma,\delta}(z)} u_0(\mathfrak{c}_{a,\gamma,\delta}(\zeta)) \frac{d\zeta}{\zeta}$$

for $z \in \mathcal{D}_{a,\gamma,\delta}$.

Theorem (*Solution*)

- *If* $0 < \mu < \dfrac{1}{p}$, *then the Dirichlet problem has a solution in* $\Re E^p(\mathcal{D}_{a,\gamma,\delta}, w)$ *if and only if*

$$\frac{1}{2\pi}\int_0^{2\pi} u_0(\mathfrak{c}_{a,\gamma,\delta}(e^{\imath\varphi}))\,\cot\frac{\varphi}{2}\,d\varphi = 0.$$

Under this condition, the solution of the Dirichlet problem in $\Re E^p(\mathcal{D}_{a,\gamma,\delta}, w)$ *has the form*

$$u(z) = \Re\int_{\mathbb{T}} \frac{1}{\pi\imath}\left(\sqrt[p]{\frac{(\mathfrak{c}^{-1}_{a,\gamma,\delta})'(z)}{(\mathfrak{c}^{-1}_{a,\gamma,\delta})'(z_0)}} - w(\mathfrak{c}^{-1}_{a,\gamma,\delta}(z))\frac{1-\mathfrak{c}^{-1}_{a,\gamma,\delta}(z)}{\zeta-\mathfrak{c}^{-1}_{a,\gamma,\delta}(z)}\zeta\right) \cdot \frac{u_0(\mathfrak{c}_{a,\gamma,\delta}(\zeta))}{\zeta-1}\,\frac{d\zeta}{\mathfrak{c}^{-1}_{a,\gamma,\delta}(z)}$$

for $z \in \mathcal{D}_{a,\gamma,\delta}$, *where* $z_0 = \mathfrak{c}_{a,\gamma,\delta}(0)$.

Remark

In the case $\mu = 0$ the operator $T(a_{a,\gamma,\delta})$ is not Fredholm. So it is reason to introducer the weight spaces.