Spectral theory of large Wiener–Hopf operators with complex–symmetric kernels and rational symbols

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A truncated Wiener–Hopf operator is of the form

$$(K_\tau f)(t) := f(t) + \int_0^\tau \mathrm{k}(t-s)f(s)ds, \qquad t \in (0,\tau). \tag{1}$$

We suppose that k is a function in $L^2(\mathbb{R})$, so that the integral operator in (1) is a Hilbert–Schmidt operator and thus compact on $L^2(0,\tau)$ for all $\tau > 0$. Let $\operatorname{sp} K_\tau$ be the spectrum of K_τ. Since $K_\tau - I$ is compact, all points in $\operatorname{sp} K_\tau \setminus \{1\}$ are eigenvalues. We are interested in the location and the asymptotic behaviour of these eigenvalues as τ tends to infinity.

The two basic assumptions stipulated in this paper are that the kernel $\mathrm{k}(t-s)$ is complex-symmetric, which means that k is a complex–valued function satisfying $\mathrm{k}(t) = \mathrm{k}(-t)$ for all $t \in \mathbb{R}$, and that the so-called *symbol* of the operator,

$$a(x) := 1 + \int_{-\infty}^{\infty} \mathrm{k}(t)\, e^{\mathrm{i}xt} dt, \quad x \in \mathbb{R},$$

is a rational function.

These two assumptions are equivalent to the requirement that

$$\mathrm{k}(t) = \begin{cases} \sum_{\ell=1}^{m} p_\ell(t)\mathrm{e}^{-\lambda_\ell t} & \text{for} \quad t > 0, \\ \sum_{\ell=1}^{m} p_\ell(-t)\mathrm{e}^{\lambda_\ell t} & \text{for} \quad t < 0, \end{cases}$$

where λ_ℓ are complex numbers with $\mathrm{Re}\,\lambda_\ell > 0$ and $p_\ell(t)$ are polynomials with complex coefficients. As $\mathrm{k}(t) = \mathrm{k}(-t)$ for all $t \in \mathbb{R}$ if and only if $a(x) = a(-x)$ for all $x \in \mathbb{R}$, the Wiener–Hopf operators considered here are just those with even rational symbols. Moreover, $\mathrm{k} \in L_2(\mathbb{R})$ implies that $\lim_{|x|\to\infty} a(x) = 1$. Therefore we may write

$$a(x) = \prod_{j=1}^{r} \frac{x^2 - \zeta_j^2}{x^2 + \mu_j^2}, \quad x \in \mathbb{R}, \tag{2}$$

where $\zeta_j \in \mathbb{C}$, $\mu_j \in \mathbb{C}$, $\mathrm{Re}\,\mu_j > 0$, and $-\zeta_j^2 \neq \mu_k^2$ for all j, k. To indicate the dependence of K_τ on the symbol a and in accordance with the literature, we henceforth denote K_τ by $W_\tau(a)$.

This work was motivated by numerous papers dedicated to Fox–Li operators. The Fox–Li operator is (1) with the kernel $\mathrm{k}(t) = \mathrm{e}^{\mathrm{i}t^2}$. Clearly, $\mathrm{k}(t-s)$ is complex-symmetric, and although the function k is not in $L^2(\mathbb{R})$, the Fox–Li operator can be shown to be the identity plus a trace class operator. Its symbol is

$$a(x) = 1 + \sqrt{\pi}\,\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}x^2/4}. \tag{3}$$

Numerical computations and arguments from physics indicate that the eigenvalues of $W_\tau(a)$ line up along a spiral commencing near the point $1 + \sqrt{\pi/\tau}\,\mathrm{e}^{\mathrm{i}\pi/4}$ and rotating clockwise to the point 1. However mathematically rigorous and at the same time satisfactory results are very sparse. These include Henry Landau's analysis of the pseudo-eigenvalues of the Fox–Li operator and Henry Landau and Harold Widom's paper, 1980 provides deep insight into the singular values of the Fox–Li operator. Of course, (3) is far from being a rational function, but we think that exploring the case of even rational symbols might well be a first step towards gaining an understanding of the situation for the Fox–Li symbol (3).

We extend a from the real line $\mathbb{R}$ to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ by defining $a(\pm\infty) := 1$. Let $\mathcal{R}(a) := a(\overline{\mathbb{R}})$. By our assumptions, $\mathcal{R}(a)$ is an analytic curve in the plane such that when x moves from $-\infty$ to 0, the symbol $a(x)$ traces out this curve from 1 to $a(0)$, and when x goes further from 0 to ∞ then $a(x)$ follows this curve back from $a(0)$ to 1. The winding number of the function a about any point outside $\mathcal{R}(a)$ is zero. Classical results on the finite section method for Wiener–Hopf operators show that if $U \subset \mathbb{C}$ is any open neighbourhood of $\mathcal{R}(a)$, then $\operatorname{sp} W_\tau(a) \subset U$ for all sufficiently large τ. In A. Böttcher and H. Widom, 1994 the limit of $\operatorname{sp} W_\tau(a)$ in the Hausdorff metric was determined for arbitrary rational symbols a. In the case of even rational symbols, the result of this paper implies that $\operatorname{sp} W_\tau(a)$ converges to $\mathcal{R}(a)$ in the Hausdorff metric.

The results of this paper are different from those cited in the previous paragraph. We provide here asymptotic expansions for individual eigenvalues. Under additional hypotheses, namely that the set $\mathcal{R}(a)$ is a curve without self-intersections and that the roots of certain polynomials are all simple, we prove the following. We associate a number $\beta > 0$ with a, consider the half-stripe

$$S_\tau := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0, \quad |\mathrm{Im}\, z| \leq \beta/\tau\},$$

and show that, for τ large enough, all the eigenvalues of $W_\tau(a)$ are contained in $a(S_\tau)$.

We consider the segments

$$I_{k,\tau} := \left[\left(k - \frac{1}{2}\right)\frac{\pi}{\tau}, \left(k + \frac{1}{2}\right)\frac{\pi}{\tau}\right]$$

In this way we obtain family of rectangles

$$S_{k,\tau} := \{z \in S_\tau : \mathrm{Re}\, z \in I_{k,\tau}\}.$$

We prove that if τ is sufficiently large then each set $a(S_{k,\tau})$ contains exactly one eigenvalue, and the eigenvalue $\lambda_{k,\tau}$ in $a(S_{k,\tau})$ has an asymptotic expansion

$$\lambda_{k,\tau} \sim a(k\pi/\tau) + \frac{c_1(k\pi/\tau)}{2\mathrm{i}\tau} + \frac{c_2(k\pi/\tau)}{(2\mathrm{i}\tau)^2} + \dots$$

with computable coefficients $c_1(k\pi/\tau), c_2(k\pi/\tau), \dots$. We also show that eigensubspaces are all one-dimensional and describe the structure of the eigenfunctions.

The Wiener–Hopf determinant

Let $U \subset \mathbb{C}$ be a sufficiently small open neighbourhood of $\mathcal{R}(a)$ and take a point $\lambda \in U \setminus \{a(0), 1\}$ such that the roots $\omega_2(\lambda), \ldots, \omega_r(\lambda)$ are all distinct. We then have

$$\frac{a(x) - \lambda}{1 - \lambda} = \frac{(x - \xi_1(\lambda)) \ldots (x - \xi_{2r}(\lambda))}{(x^2 + \mu_1^2) \ldots (x^2 + \mu_r^2)} = \prod_{j=1}^{r} \frac{x^2 - \omega_j(\lambda)^2}{x^2 + \mu_j^2}.$$

Thus, $\xi_1(\lambda), \ldots, \xi_{2r}(\lambda)$ are simply the roots $\pm\omega_1(\lambda), \ldots, \pm\omega_r(\lambda)$ labelled in a different manner.

A. Böttcher, 1989:

$$\det W_\tau\left(\frac{a-\lambda}{1-\lambda}\right) = \mathrm{e}^{\kappa\tau}\sum_M W_M \mathrm{e}^{w_M\tau} \tag{4}$$

where $\kappa = \kappa(\lambda)$ is some constant, the sum is over all subsets $M \subset \{\xi_1, \ldots, \xi_{2r}\}$ of cardinality r, and, with $M^c := \{\xi_1, \ldots, \xi_{2r}\} \setminus M$ and $R := \{\mu_1, \ldots, \mu_r\}$,

$$w_M := \sum_{\xi_j \in M^c} \mathrm{i}\xi_j,$$

$$W_M := \frac{\prod_{\xi_j\in M^c,\mu_m\in R}(\mathrm{i}\xi_j+\mu_m)\prod_{\mu_\ell\in R,\xi_k\in M}(\mu_\ell-\mathrm{i}\xi_k)}{\prod_{\mu_\ell\in R,\mu_m\in R}(\mu_\ell+\mu_m)\prod_{\xi_j\in M^c,\xi_k\in M}(\mathrm{i}\xi_j-\mathrm{i}\xi_k)}.$$

The point λ belongs to $\operatorname{sp} W_\tau(a)$ if and only if (4) is zero, whereby its algebraic multiplicity is its multiplicity as a zero of (4).

The dominant terms in (4) are those for which

$$\operatorname{Im} w_M = \sum_{\xi_j \in M^c} \operatorname{Im} \xi_j \tag{5}$$

is minimal.

The two candidates for sets M with minimal values (5) are given by

$$M_1^c := \{-\omega_1, -\omega_2, \ldots, -\omega_r\}, \quad M_2^c := \{\omega_1, -\omega_2, \ldots, -\omega_r\},$$

$$\mathrm{e}^{-\kappa\tau} \det W_\tau \left(\frac{a-\lambda}{1-\lambda}\right) = W_{M_1} \mathrm{e}^{w_{M_1}\tau} + W_{M_2} \mathrm{e}^{w_{M_2}\tau} + \sum_{M \neq M_1, M_2} W_M \mathrm{e}^{w_M \tau}, \tag{6}$$

Fix an open neighborhood $U \subset \mathbb{C}$ of $\mathcal{R}(a)$. Then $\operatorname{sp} W_\tau(a) \subset U$ for all sufficiently large τ. Let $\Pi = \{z \in \mathbb{C} : |\operatorname{Im} z| < \delta, a(z) \in U\}$. For $z \in \Pi$ consider the two functions

$$\begin{aligned} Q(z) &:= \prod_{\ell=1}^{r} (z - \mathrm{i}\mu_\ell), \\ P(z) &:= \prod_{\ell=2}^{r} [z - \omega_\ell(a(z))] \end{aligned}$$

and set

$$b(z) := \frac{Q(-z)^2}{Q(z)^2} \cdot \frac{P(z)^2}{P(-z)^2}.$$

It be $\lambda = a(z)$ with $z = \omega_1 = \omega_1(\lambda)$ in Π. The equation $\det W_\tau((a-\lambda)/(1-\lambda)) = 0$ may be written in the form

$$\mathrm{e}^{2\mathrm{i}\tau z} = b(z)(1+\varphi_\tau(z)) \tag{7}$$

where

$$\varphi_\tau(z) = \sum_{M \neq M_1, M_2} W_{M_1}^{-1} W_M \mathrm{e}^{(w_M - w_{M_1})\tau}.$$

Lemma

If $0 \notin \Pi$ *or if* $0 \in \Pi$ *but the roots* $\omega_2(a(0)), \dots, \omega_r(a(0))$ *are distinct, then*

$$\varphi_\tau(z) = O(\mathrm{e}^{-2\delta\tau}) \quad \textit{and} \quad \varphi'_\tau(z) = O(\tau\mathrm{e}^{-2\delta\tau})$$

uniformly in $z \in \Pi$.

Main results

Theorem (1)

Let $\operatorname{clos} I$ *be the closure of* I *in* $[0, \infty]$ *and suppose that for* λ *in* $a(\operatorname{clos} I)$ *the roots* $\omega_2(\lambda), \ldots, \omega_r(\lambda)$ *are distinct. Then there exists a* τ_0 *such that the following is true for every* $\tau > \tau_0$*.*

(a) *If* $\lambda = a(z) \in U$ *is an eigenvalue of* $W_\tau(a)$ *such that* $\operatorname{Re} z \in I_{k,\tau}$ *for some* $k \in \mathcal{K}_\tau(I)$*, then* $z \in S_{k,\tau}$*.*

(b) *For each* $k \in \mathcal{K}_\tau(I)$*, the set* $a(S_{k,\tau})$ *contains exactly one eigenvalue* $\lambda_{k,\tau}$ *of the operator* $W_\tau(a)$*. The algebraic multiplicity of this eigenvalue is* 1*.*

Theorem (1)

(c) *The function*

$$\Phi_{k,\tau}(z) := \frac{k\pi}{\tau} + \frac{1}{2\mathrm{i}\tau}\log b(z)$$

is a contractive map of $S_{k,\tau}$ *into itself and, letting*

$$z_{k,\tau}^{(0)} := \frac{k\pi}{\tau}, \quad z_{k,\tau}^{(n)} := \Phi_{k,\tau}(z_{k,\tau}^{(n-1)}) \qquad (n \geq 1),$$

we have

$$\lambda_{k,\tau} = a(z_{k,\tau}^{(n)}) + O(1/\tau^{n+1}) \text{ as } \tau \to \infty$$

uniformly in $k \in \mathcal{K}_\tau(I)$*, that is, there exist constants* $C_n < \infty$ *independent of* k *and* τ *such that*

$$|\lambda_{k,\tau} - a(z_{k,\tau}^{(n)})| \leq C_n/\tau^{n+1}$$

for all $\tau > \tau_0$ *and all* $k \in \mathcal{K}_\tau(I)$.

Corollary

If the points $\omega_2(1), \ldots, \omega_r(1)$ are distinct then $W_\tau(a)$ has infinitely many eigenvalues for every sufficiently large τ.

The first three iterations in Theorem 1(c) give for $\lambda_{k,\tau}$

$$a(z_0) + \frac{1}{2i\tau} a'(z_0)c_1(z_0) + \frac{1}{(2i\tau)^2}\left[a'(z_0)c_2(z_0) + \frac{a''(z_0)}{2}c_1(z_0)^2\right]$$
$$+\frac{1}{(2i\tau)^3}\left[a'(z_0)c_3(z_0) + a''(z_0)c_1(z_0)c_2(z_0) + \frac{a'''(z_0)}{6}c_1(z_0)^3\right] + O\left(\frac{1}{\tau^4}\right),$$

where

$$c_1(z_0) = \log b(z_0), \qquad c_2(z_0) = \frac{b'(z_0)}{b(z_0)}\log b(z_0),$$

$$c_3(z_0) = \frac{b'(z_0)^2}{b(z_0)^2}\log b(z_0) + \frac{b''(z_0)b(z_0) - b'(z_0)^2}{2b(z_0)^2}(\log b(z_0))^2.$$

If a is real valued, which occurs if and only if $\mathrm{k}(t) = \overline{\mathrm{k}(-t)}$ for all t, then $W_\tau(a)$ is a selfadjoint operator. In this case $|b(x)| = 1$ for $x \in \mathbb{R}$, hence the function $\Phi_{k,\tau}$ in Theorem 2(c) maps $I_{k,\tau}$ into itself and becomes

$$\Phi_{k,\tau}(x) = \frac{k\pi}{\tau} + \frac{1}{2\tau} \arg b(x)$$

for $x \in I_{k,\tau}$. It follows in particular that all eigenvalues are real, as they should be for a selfadjoint operator.

Eigenfunctions

Theorem (2)

Suppose that the numbers $\mu_1, \dots, \mu_r$ *are distinct. Let* λ *be an eigenvalue of* $W_\tau(a)$ *and assume that the roots* $\omega_2(\lambda), \dots, \omega_r(\lambda)$ *are distinct. Then every eigenfunction* $\varphi_\tau \in L^2(0,\tau)$ *of* $W_\tau(a)$ *corresponding to* λ *is of the form*

$$\varphi_\tau(t) = \sum_{j=1}^{r} \left[c_j \mathrm{e}^{\mathrm{i}\omega_j(\lambda)t} + c_{r+j} \mathrm{e}^{-\mathrm{i}\omega_j(\lambda)t} \right], \tag{8}$$

satisfies $\varphi_\tau(\tau - t) = \theta \varphi_\tau(t)$ *for all* $t \in (0,\tau)$ *with* $\theta \in \{\pm 1\}$*,*

Theorem (2)

and can be rewritten in the form

$$\varphi_\tau(t) = \begin{cases} \sum_{j=1}^{r} 2c_j \mathrm{e}^{\mathrm{i}\omega_j(\lambda)\tau/2} \cos\left(\omega_j(\lambda)\left(t - \frac{\tau}{2}\right)\right) & \text{for} \quad \theta = 1, \\ \sum_{j=1}^{r} 2\mathrm{i}c_j \mathrm{e}^{\mathrm{i}\omega_j(\lambda)\tau/2} \sin\left(\omega_j(\lambda)\left(t - \frac{\tau}{2}\right)\right) & \text{for} \quad \theta = -1. \end{cases}$$

The coefficients c_j can be computed from the linear algebraic system.

Numerical examples

$$a(x) = \frac{-(16+68\mathrm{i}) - (10+30\mathrm{i})x^2 - (3+2\mathrm{i})x^2 + x^6}{(12+16\mathrm{i}) + (20+12\mathrm{i})x^2 + (9-4\mathrm{i})x^4 + x^6} = 1 + 2\sum_{k=1}^{3} \frac{\alpha_k \mu_k}{x^2+\mu_k^2} \quad (9)$$

where $\boldsymbol{\alpha} = [-1, -\mathrm{i}, -2]$ and $\boldsymbol{\mu} = [1, 1+\mathrm{i}, 3-\mathrm{i}]$.

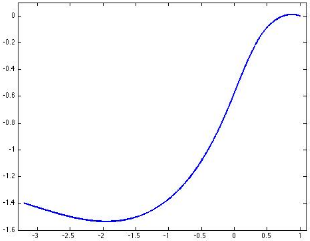

Figure: The range $\mathcal{R}(a)$ is indicated on the left, while the range of b on $(0,\infty)$ is indicated on the right. The latter is traced out clockwise, starting and terminating at 1.

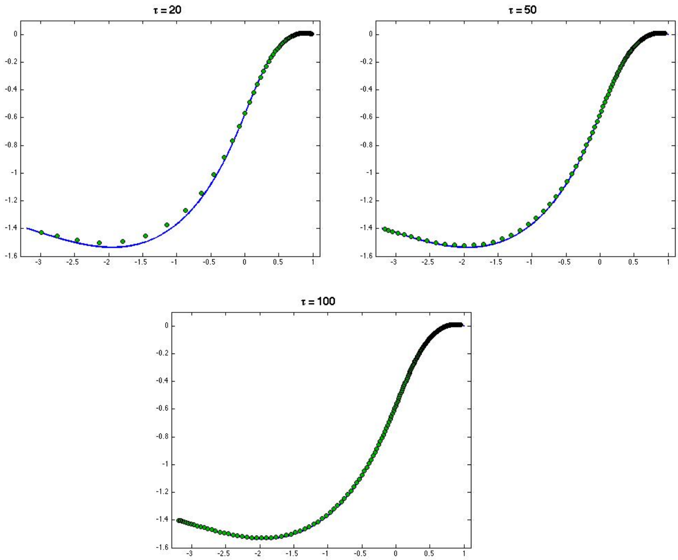

Figure: The eigenvalues, denoted by small discs and overlaid on $\mathcal{R}(a)$, for $\tau = 20, 50, 100$.

Speed of convergence

Table: The error $|z_{k,\tau}^{(n)} - \lambda_{k,\tau}|$ for $\tau = 20$, $k = 10, \dots, 17$ and the iterations $n = 0, 1, 2, 3, 4$.

	10	11	12	13	14	15	17
0	9.84_{-02}	7.69_{-02}	6.02_{-02}	4.75_{-02}	3.80_{-02}	3.10_{-02}	2.19_{-02}
1	3.69_{-03}	2.68_{-03}	1.95_{-03}	1.42_{-03}	1.03_{-03}	7.55_{-04}	4.11_{-04}
2	1.40_{-04}	9.47_{-05}	6.38_{-05}	4.26_{-05}	2.82_{-05}	1.84_{-05}	7.70_{-06}
3	5.33_{-06}	3.55_{-06}	2.09_{-06}	1.28_{-06}	7.70_{-07}	4.51_{-07}	1.44_{-07}
4	2.03_{-07}	1.18_{-07}	6.85_{-08}	3.86_{-08}	2.10_{-08}	1.10_{-08}	2.71_{-09}

Table: The error $|z_{k,\tau}^{(n)} - \lambda_{k,\tau}|$ for $\tau = 100$, $k = 50, 55, \ldots, 85$ and the iterations $n = 0, 1, 2, 3, 4$.

	50	55	60	65	70	75	85
0	2.09_{-02}	1.62_{-02}	1.26_{-02}	9.87_{-03}	7.86_{-03}	6.38_{-03}	4.48_{-03}
1	1.66_{-04}	1.19_{-04}	8.50_{-05}	6.11_{-05}	4.42_{-05}	3.21_{-05}	1.73_{-05}
2	1.32_{-06}	8.69_{-07}	5.75_{-07}	3.79_{-07}	2.48_{-07}	1.62_{-07}	6.72_{-08}
3	1.05_{-08}	6.37_{-09}	3.89_{-09}	2.35_{-09}	1.40_{-09}	8.14_{-10}	2.60_{-10}
4	8.32_{-11}	4.67_{-11}	2.63_{-11}	1.46_{-11}	7.86_{-12}	4.10_{-12}	1.01_{-12}

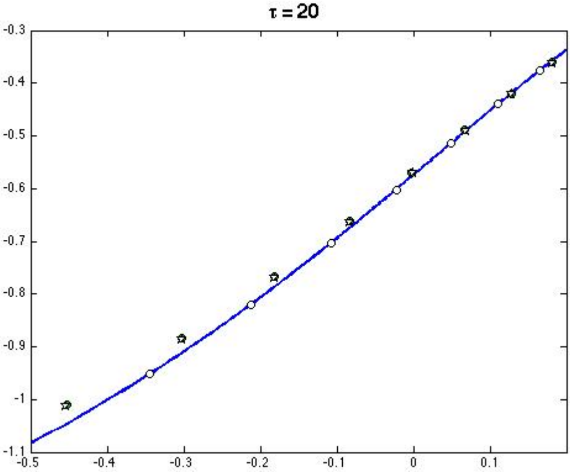

Figure: The speed of convergence of $a_{k,20}^{(n)}$ for growing n. The equispaced points $z_{k,20}^{(0)}$ are denoted by white circles, the first iteration $z_{k,20}^{(1)}$ by filled-in discs and the eigenvalues $\lambda_{k,20}^{(n)}$ by white stars.

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