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TOEPLITZ OPERATORS ON THE FOCK SPACE: RADIAL COMPONENT EFFECTS*

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The paper is devoted to the study of specific properties of Toeplitz operators with (unbounded, in general) radial symbols $a = a(r)$. Boundedness and compactness conditions, as well as examples, are given. It turns out that there exist non-zero symbols which generate zero Toeplitz operators. We characterize such symbols, as well as the class of symbols for which $T_a = 0$ implies $a(r) = 0$ a.e. For each compact set M there exists a Toeplitz operator T_a such that $\operatorname{sp} T_a = \text{ess-sp}\, T_a = M$. We show that the set of symbols which generate bounded Toeplitz operators no longer forms an algebra under pointwise multiplication.

Besides the algebra of Toeplitz operators we consider the algebra of Weyl pseudodifferential operators obtained from Toeplitz ones by means of the Bargmann transform. Rewriting our Toeplitz and Weyl pseudodifferential operators in terms of the Wick symbols we come to their spectral decompositions.

1 Introduction

Let $L_2(\mathbb{C}, d\mu)$ be the Hilbert space of square-integrable functions on $\mathbb{C}$ with the Gaussian measure

$$d\mu(z) = \pi^{-1} e^{-z\cdot\overline{z}} dv(z),$$

where $dv(z) = dxdy$ is the usual Lebesgue plane measure on $\mathbb{C} = \mathbb{R}^2$. The Fock [4, 9] (or Segal–Bargmann [2, 18]) space $F^2(\mathbb{C})$ is the subspace of $L_2(\mathbb{C}, d\mu)$ consisting of all analytic functions in $\mathbb{C}$. Denote by P the orthogonal Bargmann projection of $L_2(\mathbb{C}, d\mu)$ onto the Fock space $F^2(\mathbb{C})$. Given function $a = a(z)$, the Toeplitz operator T_a with the symbol a is defined as follows

$$T_a : \varphi \in F^2(\mathbb{C}) \longmapsto Pa\varphi \in F^2(\mathbb{C}).$$

Toeplitz operators on the Fock space have been studied intensively last years. We mention, for example, the following papers [6, 7, 8, 16, 15, 17, 19, 21].

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The present paper is devoted to the study of specific properties of Toeplitz operators with pure radial symbols $a = a(r)$, with $r = |z|$. Note that for bounded symbols $a(r)$ having limit at infinity $a(\infty) = \lim_{r\to\infty} a(r)$ corresponding Toeplitz operators are quite trivial, nothing but compact perturbations of the scalar operator $T_{a(r)} = a(\infty)I + K$.

At the same time the theory becomes very interesting and rich for symbols having irregular behavior and even being unbounded near infinity. A key feature of Toeplitz operators with radial symbols is that they are unitary equivalent to multiplication operators, more precisely: *given symbol $a = a(r)$, the Toeplitz operator T_a is unitary equivalent to the multiplication operator $\gamma_a I$, acting on one-sided l_2. The sequence $\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$, where $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, is given by*

$$\gamma_a(n) = \frac{1}{n!}\int_{\mathbb{R}_+} a(\sqrt{r})\, r^n e^{-r} dr, \quad n \in \mathbb{Z}_+. \tag{1.1}$$

Thus to guarantee the existence of the above integrals, it is natural to consider the class of symbols denoted in the paper by $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, which consists of all measurable functions $a(r)$ on $\mathbb{R}_+$ for which the following integrals are finite:

$$\int_{\mathbb{R}_+} |a(r)|\, e^{-r^2}\, r^n\, dr < \infty, \qquad n \in \mathbb{Z}_+.$$

A number of conditions which guarantee the boundedness or compactness of Toeplitz operators with $L_1^\infty(\mathbb{R}_+, e^{-r^2})$-symbols, as well as corresponding examples are given. In particular, we give an example of a symbol *unbounded at infinity* for with the corresponding Toeplitz operator is *compact*.

An interesting and unpredictable feature of our symbols is that there exist *non-zero symbols* which generate *zero Toeplitz operators*. We characterize such symbols, as well as the class of symbols for which the uniqueness theorem holds, i.e., if $T_a = 0$, then $a(r) = 0$ a.e.

Given a symbol $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$, it is clear that the Toeplitz operator T_a is bounded if and only if the corresponding sequence (1.1) belongs to l_∞. Surprisingly it turns out that *every* l_∞ - sequence is originated by a Toeplitz operator with $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ - symbol. This leads to a number of important consequences. First, for *each* compact set M there exists a Toeplitz operator T_a such that $\operatorname{sp} T_a = \operatorname{ess-sp} T_a = M$. Moreover one can *predefine* the system of eigenvalues for a Toeplitz operator. The C^*-algebra generated by bounded Toeplitz operators with $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ - symbols is *commutative* and consists *only of Toeplitz operators* with $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ - symbols; contrary to commonly known cases when starting with Toeplitz operators and generating a C^*-algebra one normally gets more complicated operators. At the same time the set of $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ - symbols which generate bounded Toeplitz operators neither forms an algebra (under pointwise multiplication), nor admits any natural norm. We give an example of two symbols which generate *bounded* Toeplitz operators, yet the Toeplitz operator associated with the product of these symbols is *unbounded*.

Besides the algebra of Toeplitz operators we consider the algebra of Weyl pseudodifferential operators obtained from the Toeplitz ones by mean of the Bargmann transform. Both algebras are commutative, for both types of operators we calculate their Wick (or Berezin) symbols, which appear to be radial as well. It is worth mentioning that rewriting our Toeplitz and Weyl pseudodifferential operators in terms of the Wick symbols we come to the spectral decomposition of operators. In a sense the Toeplitz and pseudodifferential operators under consideration are functions of the harmonic oscillator, written in Toeplitz or differential form, correspondingly.

2 Fock space over $\mathbb{C}$

Consider the space $L_2(\mathbb{C}, d\mu)$ of square-integrable functions on $\mathbb{C}$ with the Gaussian measure

$$d\mu(z) = \pi^{-1}\, e^{-z\cdot\overline{z}} dv(z),$$

where $dv(z) = dxdy$ is the usual Lebesgue plane measure on $\mathbb{C} = \mathbb{R}^2$, and its Fock [4, 9] (or Segal–Bargmann [2, 18]) subspace $F^2(\mathbb{C})$, consisting of all analytic functions in $\mathbb{C}$. The orthogonal Bargmann projection

$$P : L_2(\mathbb{C}, d\mu) \to F^2(\mathbb{C})$$

is given by the formula [4]

$$(P\varphi)(z) = \int_{\mathbb{C}} \varphi(\zeta) e^{\overline{\zeta}\cdot z} d\mu(\zeta).$$

The Fock space $F^2(\mathbb{C})$ can be described alternatively as the closure in $L_2(\mathbb{C}, d\mu)$ of the set of all smooth functions satisfying the equation

$$\frac{\partial}{\partial \overline{z}}\, \varphi = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \varphi = 0,$$

where $z = x + iy$.

Introduce the unitary operator

$$U_1 : L_2(\mathbb{C}, d\mu) \to L_2(\mathbb{R}^2) = L_2(\mathbb{R}^2, dxdy),$$

by the rule

$$(U_1\varphi)(z) = \pi^{-\frac{1}{2}} e^{-\frac{z\cdot\overline{z}}{2}} \varphi(z),$$

or

$$(U_1\varphi)(x, y) = \pi^{-\frac{1}{2}} e^{-\frac{x^2+y^2}{2}} \varphi(x + iy).$$

Then the image $F^{(1)} = U_1(F^2(\mathbb{C}))$ of the Fock space $F^2(\mathbb{C})$ is the closure of the set of all smooth functions in $L_2(\mathbb{R}^2)$ which satisfy the equation

$$\begin{aligned} D^{(1)} f = U_1 \frac{\partial}{\partial \overline{z}} U_1^{-1} f &= \left(\frac{\partial}{\partial \overline{z}} + \frac{z}{2}\right) f \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + x + iy\right) f = 0. \end{aligned}$$

Passing to polar coordinates in $\mathbb{R}^2$ we have

$$\begin{aligned}L_2(\mathbb{R}^2) = L_2(\mathbb{R}^2, dxdy) &= L_2(\mathbb{R}_+, rdr) \otimes L_2([0, 2\pi), d\alpha) \\ &= L_2(\mathbb{R}_+, rdr) \otimes L_2(S^1, \frac{dt}{it}) = L_2(\mathbb{R}_+, rdr) \otimes L_2(S^1),\end{aligned}$$

where S^1 is the unit circle, and

$$\frac{dt}{it} = |dt| = d\alpha$$

is the element of length; in addition

$$\frac{\partial}{\partial \overline{z}} + \frac{z}{2} = \frac{\cos\alpha + i\sin\alpha}{2}\left(\frac{\partial}{\partial r} + i\frac{1}{r}\frac{\partial}{\partial \alpha} + r\right) = \frac{t}{2}\left(\frac{\partial}{\partial r} - \frac{t}{r}\frac{\partial}{\partial t} + r\right).$$

Introduce the unitary operator

$$U_2 = I \otimes \mathcal{F} : \ L_2(\mathbb{R}_+, rdr) \otimes L_2(S^1) \longrightarrow L_2(\mathbb{R}_+, rdr) \otimes l_2 = l_2(L_2(\mathbb{R}_+, rdr)),$$

where the discrete Fourier transform $\mathcal{F} : L_2(S^1) \to l_2$ is given by

$$\mathcal{F} : \ f \longmapsto c_n = \frac{1}{\sqrt{2\pi}} \int_{S^1} f(t)\, t^{-n}\, \frac{dt}{it}, \quad n \in \mathbb{Z}, \tag{2.1}$$

and its inverse $\mathcal{F}^{-1} = \mathcal{F}^* : l_2 \to L_2(S^1)$ is given by

$$\mathcal{F}^{-1} : \ \{c_n\}_{n\in\mathbb{Z}} \longmapsto f = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n\, t^n.$$

Calculate

$$\begin{aligned}(I \otimes \mathcal{F})\frac{t}{2}\left(\frac{\partial}{\partial r} - \frac{t}{r}\frac{\partial}{\partial t} + r\right)(I \otimes \mathcal{F}^{-1}) \quad &: \quad \{c_n(r)\}_{n\in\mathbb{Z}} \longmapsto \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n(r)\, t^n \\ &\longmapsto \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \frac{t}{2}\left(\frac{\partial}{\partial r} - \frac{n}{r} + r\right) c_n(r)\, t^n \\ &\longmapsto \{d_n\}_{n\in\mathbb{Z}} = \left\{\frac{1}{2}\left(\frac{\partial}{\partial r} - \frac{n-1}{r} + r\right) c_{n-1}(r)\right\}_{n\in\mathbb{Z}},\end{aligned}$$

or

$$(I \otimes \mathcal{F})\frac{t}{2}\left(\frac{\partial}{\partial r} - \frac{t}{r}\frac{\partial}{\partial t} + r\right)(I \otimes \mathcal{F}^{-1})\{c_n(r)\}_{n\in\mathbb{Z}} = \left\{\frac{1}{2}\left(\frac{\partial}{\partial r} - \frac{n-1}{r} + r\right) c_{n-1}(r)\right\}_{n\in\mathbb{Z}}.$$

Thus the image $F^{(2)} = U_2(F^{(2)})$ of the space $F^{(1)})$ can be described as the subspace of $L_2(\mathbb{R}_+, rdr) \otimes l_2 = l_2(L_2(\mathbb{R}_+, rdr))$ which is the closure of all sequences $\{c_n(r)\}_{n\in\mathbb{Z}}$ with smooth components satisfying the equations

$$\frac{1}{2}\left(\frac{\partial}{\partial r} - \frac{n}{r} + r\right) c_n(r) = 0, \quad n \in \mathbb{Z}. \tag{2.2}$$

The equations (2.2) are easy to solve, and their general solutions have the form

$$c_n(r) = c_n' \, r^n \, e^{-\frac{r^2}{2}} = c_n \sqrt{\frac{2}{|n|!}} \, r^n \, e^{-\frac{r^2}{2}}, \quad n \in \mathbb{Z}.$$

But each function $c_n(r)$ has to be in $L_2(\mathbb{R}_+, rdr)$, which implies that $c_n(r) \equiv 0$, for each $n < 0$. Thus the space $F^{(2)}$ ($\subset L_2(\mathbb{R}_+, rdr) \otimes l_2 = l_2(L_2(\mathbb{R}_+, rdr))$) coincides with the space of all two-sided sequences $\{c_n(r)\}_{n\in\mathbb{Z}}$ with

$$c_n(r) = \begin{cases} c_n \sqrt{\frac{2}{n!}} \, r^n \, e^{-\frac{r^2}{2}}, & \text{if } \ n \in \mathbb{Z}_+ \\ 0, & \text{if } \ n \in \mathbb{Z}_- \end{cases},$$

where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$, $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{Z}_+$, and

$$\|\{c_n(r)\}_{n\in\mathbb{Z}}\| = \left(\sum_{n\in\mathbb{Z}_+} |c_n|^2 \right)^{1/2} = \|\{c_n\}_{n\in\mathbb{Z}_+}\|_{l_2}.$$

For each $n \in \mathbb{Z}_+$ introduce the unitary operator

$$u_n : L_2(\mathbb{R}_+) = L_2(\mathbb{R}_+, dr) \longrightarrow L_2(\mathbb{R}_+, rdr)$$

by the rule

$$(u_n f)(r) = \omega_n(r) f(\alpha_n(r)),$$

where

$$\begin{aligned} \omega_n(r) &= \sqrt{\frac{2}{n!}} \, r^n \left(\sum_{k=0}^{n} \frac{r^{2k}}{k!} \right)^{-\frac{1}{2}}, \\ \alpha_n(r) &= r^2 - \ln \sum_{k=0}^{n} \frac{r^{2k}}{k!} \, . \end{aligned} \tag{2.3}$$

Finally, define the unitary operator

$$U_3 : l_2(L_2(\mathbb{R}_+, rdr)) \longrightarrow l_2(L_2(\mathbb{R}_+)) = L_2(\mathbb{R}_+) \otimes l_2$$

as follows

$$U_3 : \{c_n(r)\}_{n\in\mathbb{Z}} \longmapsto \{(u_{|n|}^{-1} c_n)(r)\}_{n\in\mathbb{Z}}.$$

Then the space $F^{(3)} = U_3(F^{(2)})$ coincides with the space of all sequences $\{d_n(r)\}_{n\in\mathbb{Z}}$, where

$$d_n = u_n^{-1} \left(c_n \sqrt{\frac{2}{n!}} \, r^n \, e^{-\frac{r^2}{2}} \right) = c_n e^{-\frac{r}{2}},$$

for $n \in \mathbb{Z}_+$, and $d_n(r) \equiv 0$, for $n \in \mathbb{Z}_-$.

We introduce some notation. Let $\ell_0(r) = e^{-\frac{r}{2}}$; we have $\ell_0(r) \in L_2(\mathbb{R}_+)$ and $\|\ell_0(r)\| = 1$. Denote by L_0 the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_0(r)$, then the one-dimensional projection P_0 of $L_2(\mathbb{R}_+)$ onto L_0 has the form

$$(P_0 f)(r) = \langle f, \ell_0 \rangle \cdot \ell_0 = \int_{\mathbb{R}_+} f(\rho)\, e^{-\frac{r+\rho}{2}}\, d\rho. \tag{2.4}$$

Denote by l_2^+ the subspace of (two-sided) l_2, consisting of all sequences $\{c_n\}_{n\in\mathbb{Z}}$, such that $c_n = 0$ for all $n \in \mathbb{Z}_-$, and denote by p^+ the orthogonal projection of l_2 onto l_2^+. Introduce the sequences $\chi_+ = \{\chi_+(n)\}_{n\in\mathbb{Z}} \in l_\infty$, where $\chi_+(n) = 1$ for $n \in \mathbb{Z}_+$, and $\chi_+(n) = 0$ for $n \in \mathbb{Z}_-$. Then obviously $p^+ = \chi_+ I$.

Now $F^{(3)} = L_0 \otimes l_2^+$, and the orthogonal projection $P^{(3)}$ of $l_2(L_2(\mathbb{R}_+)) = L_2(\mathbb{R}_+) \otimes l_2$ onto $F^{(3)}$ has obviously the form

$$P^{(3)} = P_0 \otimes p^+.$$

The above work leads to the following theorem.

Theorem 2.1 *The unitary operator $U = U_3 U_2 U_1$ is an isometric isomorphism of the space $L_2(\mathbb{C}, d\mu)$ onto $L_2(\mathbb{R}_+) \otimes l_2$ under which*

1. *the Fock space $F^2(\mathbb{C})$ is mapped onto $L_0 \otimes l_2^+$*

$$U : F^2(\mathbb{C}) \longrightarrow L_0 \otimes l_2^+,$$

where L_0 is the one-dimensional subspace of $L_2(\mathbb{R}_+)$, generated by $\ell_0(r) = e^{-\frac{r}{2}}$,

2. *the Bargmann projection P is unitary equivalent to*

$$U\, P\, U^{-1} = P_0 \otimes p^+,$$

where P_0 is the one-dimensional projection (2.4) of $L_2(\mathbb{R}_+)$ onto L_0.

Introduce the isometric imbedding

$$R_0 : l_2^+ \longrightarrow L_2(\mathbb{R}_+) \otimes l_2$$

by the rule

$$R_0 :\ \{c_n\}_{n\in\mathbb{Z}_+} \ \longmapsto\ \ell_0(r)\{\chi_+(n) c_n\}_{n\in\mathbb{Z}}.$$

The image of R_0 is obviously coincides with the space $F^{(3)}$. The adjoint operator $R_0^* : L_2(\mathbb{R}_+) \otimes l_2 \to l_2^+$ is given by

$$R_0^* :\ \{c_n(r)\}_{n\in\mathbb{Z}} \ \longmapsto\ \left\{\chi_+(n) \int_{\mathbb{R}_+} c_n(\rho) e^{-\frac{\rho}{2}}\, d\rho\right\}_{n\in\mathbb{Z}_+},$$

and

$$\begin{aligned} R_0^* R_0 = I \ &:\ l_2^+ \longrightarrow l_2^+ \\ R_0 R_0^* = P^{(3)} \ &:\ L_2(\mathbb{R}_+) \otimes l_2 \longrightarrow F^{(3)} = L_0 \otimes l_2^+. \end{aligned}$$

Now the operator $R = R_0^* U$ maps the space $L_2(\mathbb{C}, d\mu)$ onto l_2^+, and the restriction

$$R|_{F^2(\mathbb{C})} : F^2(\mathbb{C}) \longrightarrow l_2^+$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : l_2^+ \longrightarrow F^2(\mathbb{C}) \subset L_2(\mathbb{C}, d\mu)$$

is an isometric isomorphism of l_2^+ onto the subspace $F^2(\mathbb{C})$ of the space $L_2(\mathbb{C}, d\mu)$.

Remark 2.2 We have

$$\begin{array}{rcl} R\,R^* = I & : & l_2^+ \longrightarrow l_2^+ \\ R^* R = P & : & L_2(\mathbb{C}, d\mu) \longrightarrow F^2(\mathbb{C}). \end{array}$$

Theorem 2.3 *The isometric isomorphism*

$$R^* = U^* R_0 : l_2^+ \longrightarrow F^2(\mathbb{C})$$

is given by

$$R^* : \{c_n\}_{n\in\mathbb{Z}_+} \longmapsto \sum_{n\in\mathbb{Z}_+} \frac{c_n}{\sqrt{n!}}\, z^n.$$

PROOF. Calculate

$$\begin{aligned} R^* = U_1^* U_2^* U_3^* R_0 \quad : \quad & \{c_n\}_{n\in\mathbb{Z}_+} \longmapsto U_1^* U_2^* U_3^* \left(\{c_n\, e^{-\frac{r^2}{2}}\}_{n\in\mathbb{Z}_+}\right) \\ = \;\; & U_1^* U_2^* \left(\{c_n \sqrt{\frac{2}{n!}}\, r^n\, e^{-\frac{r^2}{2}}\}_{n\in\mathbb{Z}_+}\right) \\ = \;\; & U_1^* \left(\frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}_+} c_n \sqrt{\frac{2}{n!}}\, (rt)^n\, e^{-\frac{r^2}{2}}\right) \\ = \;\; & \sum_{n\in\mathbb{Z}_+} \frac{c_n}{\sqrt{n!}}\, z^n. \end{aligned}$$

□

Corollary 2.4 [2] *A function*

$$\varphi(z) = \sum_{n\in\mathbb{Z}_+} a_n z^n$$

belongs to the Fock space F^2 if and only if

$$\sum_{n\in\mathbb{Z}_+} |\, a_n|^2\, n! < +\infty,$$

and in this case

$$\|\, \varphi(z)\| = \left(\sum_{n\in\mathbb{Z}_+} |\, a_n|^2\, n!\right)^{\frac{1}{2}}.$$

Corollary 2.5 *The inverse isomorphism*

$$R : F^2(\mathbb{C}) \longrightarrow l_2^+$$

is given by

$$R : \varphi(z) \ \longmapsto \ \left\{ \frac{1}{\sqrt{n!}} \int_{\mathbb{C}} \varphi(z)\, \overline{z}^n \, d\mu(z) \right\}_{n\in\mathbb{Z}_+}.$$

3 Toeplitz operators with radial symbols

Denote by $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ the linear space of all measurable functions $a(r)$ on $\mathbb{R}_+$ for which the following integrals are finite

$$\int_{\mathbb{R}_+} |a(r)|\, e^{-r^2}\, r^n\, dr < \infty, \tag{3.1}$$

for all $n \in \mathbb{Z}_+$.

In this section we will study Toeplitz operators with symbols from $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, acting on the Fock space $F^2(\mathbb{C})$.

Theorem 3.1 *Let $a = a(r)$ belong to $L_1^\infty(\mathbb{R}_+, e^{-r^2})$. Then the Toeplitz operator T_a acting on the Fock space $F^2(\mathbb{C})$ is unitary equivalent to the multiplication operator $\gamma_a I$ acting on l_2^+. The sequence $\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$ is given by*

$$\gamma_a(n) = \frac{1}{n!} \int_{\mathbb{R}_+} a(\sqrt{r})\, r^n e^{-r} dr, \quad n \in \mathbb{Z}_+. \tag{3.2}$$

Proof. The operator T_a is obviously unitary equivalent to the operator

$$\begin{aligned}
R\,T_a\,R^* &= R\,PaPR^* = R(R^*R)a(R^*R)R^* \\
&= (RR^*)RaR^*(RR^*) = RaR^* \\
&= R_0^* U_3 U_2 U_1 a(r) U_1^{-1} U_2^{-1} U_3^{-1} R_0 \\
&= R_0^* U_3 (I \otimes \mathcal{F}) a(r) (I \otimes \mathcal{F}^{-1}) U_3^{-1} R_0 \\
&= R_0^* U_3 \{a(r)\} U_3^{-1} R_0 \\
&= R_0^* \{a(\alpha_{|n|}^{-1}(r))\} R_0,
\end{aligned}$$

where the function $\alpha_n(r)$ is given by (2.3). Now

$$R_0^* \{a(\alpha_{|n|}^{-1}(r))\} R_0 \{c_n\}_{n\in\mathbb{Z}^+} = \left\{ \int_{\mathbb{R}_+} a(\alpha_n^{-1}(r))\, c_n\, e^{-r}\, dr \right\}_{n\in\mathbb{Z}_+} = \{\gamma_a(n) \cdot c_n\}_{n\in\mathbb{Z}_+},$$

where

$$\begin{aligned}
\gamma_a(n) &= \int_{\mathbb{R}_+} a(\alpha_n^{-1}(r))\, e^{-r} dr = \int_{\mathbb{R}_+} a(r)\, e^{-\alpha_n(r)} \alpha_n'(r)\, dr \\
&= \frac{2}{n!} \int_{\mathbb{R}_+} a(r)\, r^{2n+1} e^{-r^2} dr = \frac{1}{n!} \int_{\mathbb{R}_+} a(\sqrt{r})\, r^n e^{-r} dr.
\end{aligned}$$

□

Theorem 3.2 *The Toeplitz operator T_a with radial symbol $a = a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ is bounded on $F^2(\mathbb{C})$ if and only if*

$$\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}_+} \in l_\infty,$$

and

$$\|T_a\| = \sup_{n\in\mathbb{Z}_+} |\gamma_a(n)|.$$

The Toeplitz operator T_a is compact if and only if

$$\lim_{n\to\infty} \gamma_a(n) = 0.$$

Proof. Follows directly from the previous theorem. □

We comment on the last two theorems. First of all a Toeplitz operator T_a with symbol $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ is a well defined linear operator (unbounded, in general) with a dense domain. In fact, the set $F_0^2(\mathbb{C})$ of all polynomials on z forms a dense subset on the Fock space. For $p(z) = \sum_{n=0}^m c_n z^n \in F_0^2(\mathbb{C})$ and $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ we have

$$\begin{aligned}
(T_a p)(z) &= \frac{1}{\pi}\int_{\mathbb{C}} a(|\xi|)p(\xi)e^{\overline{\xi}z}e^{-|\xi|^2}dv(\xi) \\
&= \frac{1}{\pi i}\int_{\mathbb{R}_+}\left(\int_{S^1}\left(\sum_{n=0}^m c_n r^n t^n\right)e^{(rz)t^{-1}}\frac{dt}{t}\right)a(r)\,e^{-r^2}rdr \\
&= \sum_{n=0}^m \frac{1}{\pi}c_n\int_{\mathbb{R}_+}\int_{S^1} t^n\left(\sum_{k=0}^\infty \frac{r^k z^k t^{-k}}{k!}\right)\frac{dt}{t}\,a(r)e^{-r^2}r^{n+1}dr \\
&= \sum_{n=0}^m c_n z^n\left(\frac{2}{n!}\int_{\mathbb{R}_+} a(r)\,r^{2n+1}e^{-r^2}dr\right) \\
&= \sum_{n=0}^m c_n z^n \gamma_a(n).
\end{aligned} \tag{3.3}$$

Thus

$$T_a p \in F_0^2(\mathbb{C}) \subset F^2(\mathbb{C}),$$

and the set $F_0^2(\mathbb{C})$ is a domain for each Toeplitz operator T_a with symbol $a(r)$ which satisfies the condition (3.1). That is, by (3.3) the operator T_a has a bounded extension to the whole space $F^2(\mathbb{C})$ if and only if the sequence $\gamma_a(n)$ is bounded.

Corollary 3.3 *The spectrum of a bounded Toeplitz operator T_a is given by*

$$\mathrm{sp}\, T_a = \overline{\{\gamma_a(n) : n \in \mathbb{Z}_+\}},$$

and its essential spectrum ess-sp T_a *coincides with the set of all limit points of the sequence* $\{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$.

For bounded symbols $a(z) \in L_\infty(\mathbb{C})$ Berger and Coburn [6] proved that $T_a = 0$ if and only if $a = 0$ almost everywhere. Folland [10], p. 140, extends this uniqueness result for the class of unbounded symbols which satisfy the inequality (in our notations)

$$|a(z)| \leq \text{const}\, e^{\delta |z|^2}, \qquad \text{for some} \qquad \delta < 1. \tag{3.4}$$

Surprisingly it turns out that for our class of symbols there exist nontrivial ones for which $T_a = 0$. Let us describe such symbols.

Denote by $\dot{\Pi}_+$ the one point compactification of the upper half plane $\Pi_+ \subset \mathbb{C}$. Introduce the class $H_1^\infty(\mathbb{R}, e^{-r^2})$ of functions $f(x)$ which admit an analytic continuation to the upper half plane Π_+, continuous on $\dot{\Pi}_+$, and which admit the following representation

$$f(x) = (F^{-1}a(r)e^{-r^2})(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} e^{irx} a(r)\, e^{-r^2} dr,$$

where $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$. Here F^{-1} is the inverse Fourier transform of a function supported on $\mathbb{R}_+$. By the condition (3.1) the function $f(z)$ which is the analytic continuation of $f(x) \in H_1^\infty(\mathbb{R}, e^{-r^2})$ tends to zero at infinity, has the derivatives of all orders which are analytic in Π_+, continuous on $\dot{\Pi}_+$, and tend to zero at infinity as well.

Let now $H_{1,0}^\infty(\mathbb{R}, e^{-r^2})$ be the subclass of $H_1^\infty(\mathbb{R}, e^{-r^2})$ which consists of all functions having the property

$$f^{(2n+1)}(0) = 0, \qquad n \in \mathbb{N}.$$

Finally, let $L_{1,0}^\infty(\mathbb{R}_+, e^{-r^2}) = e^{r^2} F(H_{1,0}^\infty(\mathbb{R}, e^{-r^2}))$.

Theorem 3.4 *For a symbol $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ the Toeplitz operator $T_a = 0$ if and only if*

$$a(r) \in L_{1,0}^\infty(\mathbb{R}_+, e^{-r^2}).$$

Proof. Follows immediately from the well known in the theory of Fourier transform fact, that

$$f^{(2n+1)}(0) = \frac{i^{2n+1}}{\sqrt{2\pi}} \int_{\mathbb{R}_+} a(r)\, e^{-r^2} r^{2n+1} dr = \frac{n!}{2} \frac{i^{2n+1}}{\sqrt{2\pi}} \gamma_a(n).$$

□

Theorem 3.5 *The class $L_{1,0}^\infty(\mathbb{R}_+, e^{-r^2})$ is not trivial, i.e., contains non identically zero functions.*

Proof. To prove the theorem we start with an example of a non trivial function from $H_{1,0}^\infty(\mathbb{R}, e^{-r^2})$. Namely, let

$$f(x) = \exp(\overline{c}_0 x^{-\rho} + c_0 x^\rho),$$

where $0 < \rho < 1$, and $c_0 = e^{i\frac{\pi}{2}(2-\rho)}$. Define the functions $z^{\pm\rho}$ as follows: for $z = |z|e^{i\varphi}$ with $-\frac{\pi}{2} \leq \varphi < \frac{3\pi}{2}$ we set

$$z^{\pm\rho} = |z|^{\pm\rho} e^{\pm\rho\varphi}.$$

Now for $\varphi \in [0, \pi]$ we have

$$\begin{aligned} f(z) &= \exp(|z|^{-\rho} \exp(-i(\frac{\pi}{2}(2-\rho) + \rho\varphi)) + |z|^{\rho} \exp(i(\frac{\pi}{2}(2-\rho) + \rho\varphi))) \\ &= \exp(-(|z|^{-\rho} + |z|^{\rho}) \cos(-\frac{\pi\rho}{2} + \rho\varphi) - i(|z|^{-\rho} - |z|^{\rho}) \sin(-\frac{\pi\rho}{2} + \rho\varphi)). \end{aligned}$$

From $0 \leq \varphi \leq \pi$ it follows that

$$-\frac{\pi}{2} < -\frac{\pi\rho}{2} \leq -\frac{\pi\rho}{2} + \rho\varphi \leq \frac{\pi\rho}{2} < \frac{\pi}{2}.$$

It is easy to see now that the function f has zero derivatives of all orders at the origin: $f^{(n)}(0) = 0$, for all $n \in \mathbb{Z}_+$. Furthermore the function $f(x)$ belongs to the class $\mathcal{S}$ of infinitely differentiable functions rapidly decreasing at infinity. The class $\mathcal{S}$ in invariant with respect to the Fourier transform, thus the function $(Ff)(r)$ belongs to $\mathcal{S}$ as well, and its support is in $\mathbb{R}_+$. Finally, $a(r) = (Ff)(r)\, e^{r^2} \in L_1^{\infty}(\mathbb{R}_+, e^{-r^2})$. $\square$

Let us describe now the subclass of symbols for which the uniqueness theorem (i.e., if $T_a = 0$, then $a = 0$ a.e.) holds. Note, that for radial symbols our class, described by the condition (3.5) bellow, is wider that one given by the Folland condition (3.4).

Given $\varepsilon > 0$, denote by $E^{\varepsilon}(\mathbb{R}_+, e^{-r^2})$ the subclass of $L_1^{\infty}(\mathbb{R}_+, e^{-r^2})$ which consists of all functions $a(r)$ satisfying at $+\infty$ the following estimate

$$|a(r)| \leq \text{const } e^{r^2 - \varepsilon r}. \tag{3.5}$$

Theorem 3.6 *For a symbol $a(r) \in E^{\varepsilon}(\mathbb{R}_+, e^{-r^2})$ the Toeplitz operator is equal to 0 if and only if $a(r) = 0$ a.e.*

Proof. Let $a(r) \in E^{\varepsilon}(\mathbb{R}_+, e^{-r^2})$. By property (3.5) the function $f(x) = (F^{-1}a(r)e^{-r^2})(x)$ admits an analytic continuation (which we will denote by $f(z)$) not only to the upper half plane Π_+, but to a larger half plane

$$\Pi_{-\varepsilon} = \{z \in \mathbb{C} : \operatorname{Im} z > -\varepsilon\}.$$

Thus the function $f(z)$ is analytic at the point $z = 0$, and in a disk of a radius less then ε can be represented as the absolutely convergent series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, z^n. \tag{3.6}$$

Let now $T_a = 0$. Then by Theorem 3.4 all odd derivatives of $f(z)$ vanish at the origin, and the representation (3.6) reduces to

$$f(z) = \sum_{n=10}^{\infty} \frac{f^{(2n)}(0)}{2n!}\, z^n. \tag{3.7}$$

Now the function $f(-z)$ is analytic in the following half plane

$$\Pi_{\varepsilon}^{-} = \{z \in \mathbb{C} : \operatorname{Im} z < \varepsilon\}.$$

The functions $f(z)$ and $f(-z)$ are definitely coincide on the open disk having radius ε (the domain of convergence of the series (3.7)), and therefore coincide in the strip

$$\{z \in \mathbb{C} : \, |\mathrm{Im}\, z| < \varepsilon\}.$$

Thus the function $f(z)$ is analytic in the whole complex plane $\mathbb{C}$, and tends to zero at infinity. By the Liouville theorem $f(z)$ has to be identically zero, which implies that $a(r) = 0$ a.e.

□

By Theorem 3.2 a bounded Toeplitz operator T_a with a radial symbol $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ is unitary equivalent to the multiplication operator $\gamma_a I$ acting on the space l_2^+, where the l_∞– sequence $\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$ is given by (3.2). The natural question appears: *how wide is the class of l_∞– sequences which are originated by Toeplitz operators with radial symbols.* The next theorem gives a complete answer to this question.

Theorem 3.7 *For each sequence $\gamma = \{\gamma(n)\}_{n\in\mathbb{Z}_+}$ in l_∞ there exists a symbol $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ such that the Toeplitz operator T_a is unitary equivalent to the multiplication operator by this sequence γ; i.e., $\gamma_a = \gamma$.*

PROOF. Consider a sequence $\gamma = \{\gamma(n)\}_{n\in\mathbb{Z}_+}$ from l_∞. Introduce the function

$$f_1(x) = \frac{1}{2\sqrt{2\pi}} \sum_{n=1}^{\infty} \gamma(n) \, \frac{(-i)^{2n+1}\, n!}{(2n+1)!} \, x^{2n+1}.$$

This series converges on the whole real line $\mathbb{R}$ (and even in $\mathbb{C}$), and

$$f_1^{(2n+1)}(0) = \frac{(-i)^{2n+1}\, n!}{2\sqrt{2\pi}} \, \gamma(n). \tag{3.8}$$

Let $\chi_c(x)$ be an infinitely smooth function supported in an interval $(-c, c)$, and identically equal to 1 on a neighborhood of the point 0. Then for the function $f_2(x) = \chi_c(x) f_1(x)$ by (3.8) we have

$$f_2^{(2n+1)}(0) = \frac{(-i)^{2n+1}\, n!}{2\sqrt{2\pi}} \, \gamma(n).$$

Represent the function $f_2(x)$ in the form $f_2(x) = f_2^+(x) + f_2^-(x)$, where $f_2^\pm(x) = (P^\pm f_2)(x)$, and $P^\pm$ are the standard analytical projections on the real line:

$$P^\pm = \frac{1}{2}(I + S_\mathbb{R}), \qquad \text{where} \qquad (S_\mathbb{R} f)(x) = \frac{1}{\pi i} \int_\mathbb{R} \frac{f(\tau)}{\tau - x} \, d\tau.$$

It is known (see, for example, [12]) that the class $\mathcal{S}$ is invariant with respect to the singular integral operator $S_\mathbb{R}$, thus the functions $f_2^\pm(x)$ belong to $\mathcal{S}$. Moreover, it is easy to see that both $f_2^+(x)$ and $f_2^-(-x)$ belong to $H_1^\infty(\mathbb{R}, e^{-r^2})$.

Introduce now the function

$$f(x) = f_2^+(x) - f_2^-(-x).$$

For its odd derivatives we obviously have

$$f^{(2n+1)}(0) = (f_2^+)^{(2n+1)}(0) + (f_2^-)^{(2n+1)}(0).$$

The property (see, for example, [11]) $(P^{\pm} f)^{(n)}(x) = (P^{\pm} f^{(n)})(x)$ implies that

$$f^{(2n+1)}(0) = f_2^{(2n+1)}(0) = \frac{(-i)^{2n+1}\, n!}{2\sqrt{2\pi}}\, \gamma(n). \tag{3.9}$$

Finally, introduce the function (symbol)

$$a(r) = e^{r^2} (Ff)(r). \tag{3.10}$$

The function $f(x)$ belongs to $H_1^\infty(\mathbb{R}, e^{-r^2})$, and thus $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$. Now the Toeplitz operator T_a with symbol (3.10) is unitary equivalent to the multiplication operator by the sequence $\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$, where

$$\begin{aligned} \gamma_a(n) &= \frac{2}{n!} \int_{\mathbb{R}_+} a(r) e^{-r^2} r^{2n+1} dr \\ &= \frac{2\sqrt{2\pi}}{n!}\, i^{2n+1} f^{(2n+1)}(0). \end{aligned}$$

Thus by (3.9) we have $\gamma_a(n) = \gamma(n)$ for all $n \in \mathbb{Z}_+$. $\square$

Remark 3.8 Note, that the class $L_{1,0}^\infty(\mathbb{R}_+, e^{-r^2})$, which generates zero Toeplitz operators, is quite large. In addition to methods of Theorem 3.6 one can construct symbols from the class $L_{1,0}^\infty(\mathbb{R}_+, e^{-r^2})$ using the ideas of the proof of Theorem 3.7. Let $f(x)$ be a function of the class $\mathcal{S}$ having all derivatives equal to zero at the origin. Then the function

$$a(r) = e^{r^2} F((P^+ f)(x) - (P^- f)(-x))(r)$$

belongs to the class $L_{1,0}^\infty(\mathbb{R}_+, e^{-r^2})$.

4 Boundedness, compactness, spectral properties

We start with conditions which guarantee the boundedness or compactness of Toeplitz operators with radial symbols from $L_1^\infty(\mathbb{R}_+, e^{-r^2})$.

Theorem 4.1 *Let $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$. Then the Toeplitz operator T_a is bounded on $F^2(\mathbb{C})$ (the corresponding sequence (3.2) is bounded) if one of the following conditions holds:*

(i) $a(r) \in L_\infty(\mathbb{R}_+)$,

(ii) the sequence $\gamma_a^{(2)} = \{\gamma_a^{(2)}(n)\}_{n\in\mathbb{Z}_+}$ is bounded, where

$$\gamma_a^{(2)}(n) = \frac{1}{n!} \int_{\mathbb{R}_+} |a(\sqrt{r})|\, r^n\, e^{-r}\, dr,$$

(iii) *the function*

$$B(r) = \int_r^{+\infty} a(\sqrt{r})\, e^{r-u} du \tag{4.1}$$

is bounded.

The Toeplitz operator T_a is compact on $F^2(\mathbb{C})$ (the corresponding sequence (3.2) tends to zero) if the one of the following conditions holds:

(i′) $\lim_{r\to+\infty} a(r) = 0$,

(ii′) $\lim_{n\to\infty} \gamma_a^{(2)}(n) = 0$,

(iii′) $\lim_{r\to+\infty} B(r) = 0$.

Proof. The condition (i) and (i') are well known, and are stated here for the completeness only. Let now the condition (ii) (or (ii')) holds. Then one obviously has

$$\begin{aligned} |\gamma_a(n)| &= \frac{1}{n!}\left|\int_{\mathbb{R}_+} a(\sqrt{r}) r^n e^{-r} dr\right| \\ &\leq \frac{1}{n!}\int_{\mathbb{R}_+} |a(\sqrt{r})| r^n e^{-r} dr = \gamma_a^{(2)}(n), \end{aligned}$$

and the statement (ii) (or (ii')) proved.

To prove the statement (iii) (or (iii')) first integrate by parts:

$$\begin{aligned} \gamma_a(n) &= -\frac{1}{n!}\int_{\mathbb{R}_+} r^n d\left(\int_r^{+\infty} a(\sqrt{u})\, e^{-u} du\right) \\ &= \frac{1}{(n-1)!}\int_{\mathbb{R}_+} B(r)\, r^{n-1}\, e^{-r} dr = \gamma_B(n-1), \end{aligned}$$

and then apply statement (i) (or (i')) to the function $B(r)$. □

Remark 4.2 Let us mention that the statement (ii) is a necessary and sufficient condition on the function $a(r)$ in order for the multiplication operator $a(r)I : F^2(\mathbb{C}) \longrightarrow L_2(\mathbb{C}, d\mu)$ to be bounded.

Let $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$, and $b(r) = a(\sqrt{r})$. Introduce the following averages of the function $b(r)$ $(a(r))$:

$$B_{(j)}(r) = \int_r^{+\infty} B_{(j-1)}(u)\, e^{r-u} dr, \qquad j = 1, 2, ...,$$

and $B_{(0)}(r) = b(r)$. Note, that the function (4.1) is just $B_{(1)}(r)$.

Theorem 4.3 *Let $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$. Then the Toeplitz operator T_a is bounded, compact or unbounded (the corresponding sequence (3.2) is bounded, tends to zero, or unbounded) if for some integer j the corresponding condition holds:*

1. $B_{(j)}(r) \in L_\infty(\mathbb{R}_+)$,

2. $\lim_{r\to\infty} B_{(j)}(r) = 0$,

3. assume in addition that

$$\operatorname{Re} B_{(j)}(r) \geq c \qquad \left(\operatorname{Im} B_{(j)}(r) \geq c\right),$$

for some $c \in \mathbb{R}$*; then the condition is*

$$\lim_{r\to\infty} \inf_{s>r} \operatorname{Re} B_{(j)}(r) = +\infty \qquad \left(\lim_{r\to\infty} \inf_{s>r} \operatorname{Im} B_{(j)}(r) = +\infty\right).$$

Proof. Statement 1) is a generalization of the statements (i) and (iii) of the previous theorem; statement 2) is a generalization of the statements (i') and (iii') correspondingly. Their proofs are analogous to those of the previous theorem, and are based on integrating by parts, j times.

To prove statement 3) we may assume that $c = 0$, otherwise consider the symbol $a(r) - c \quad (a(r) - ic)$. Integrating by parts j times we have

$$\gamma_a(n) = \frac{1}{(n-j)!} \int_{\mathbb{R}_+} B_{(j)}(r)\, r^{n-j}\, e^{-r} dr.$$

Estimate now the real part of this integral

$$\begin{aligned}
\operatorname{Re} \gamma_a(n) &\geq \frac{1}{(n-j)!} \int_{\frac{n-j}{2}}^{\infty} \operatorname{Re} B_{(j)}(r)\, r^{n-j}\, e^{-r} dr \\
&= \inf_{s>\frac{n-j}{2}} \operatorname{Re} B_{(j)}(s) \left(\frac{1}{(n-j)!} \int_0^\infty r^{n-j}\, e^{-r} dr - \frac{1}{(n-j)!} \int_0^{\frac{n-j}{2}} r^{n-j}\, e^{-r} dr \right) \\
&\geq \inf_{s>\frac{n-j}{2}} \operatorname{Re} B_{(j)}(s) \left(1 - \frac{n-j}{2} \, \frac{e^{-\frac{n-j}{2}}}{(n-j)!} \left(\frac{n-j}{2}\right)^{n-j} \right).
\end{aligned}$$

We have used here the fact that the function $r^{n-j}\, e^{-r}$ is increasing in the interval $(0, \frac{n-j}{2})$. Applying the asymptotic Euler formula for the Gamma function we have

$$\operatorname{Re} \gamma_a(n) \geq \inf_{s>\frac{n-j}{2}} \operatorname{Re} B_{(j)}(s) \left(1 - M \, \frac{e^{-\frac{n-j}{2}} \left(\frac{n-j}{2}\right)^{n-j+1}}{(n-j+1)^{n+j+\frac{1}{2}}\, e^{-(n-j+1)}} \right),$$

where the constant $M > 0$ does not depend on n. The second summand in the last formula tends to zero when $n \to \infty$, thus for sufficiently large n we have

$$\operatorname{Re} \gamma_a(n) \geq \frac{1}{2} \inf_{s>\frac{n-j}{2}} \operatorname{Re} B_{(j)}(s),$$

which by the first condition in 3) gives the unboundedness of the sequence $\{\operatorname{Re} \gamma_a(n)\}$, and thus of the sequence $\{\gamma_a(n)\}$ as well.

The proof of the second condition in 3) is quite analogous. $\square$

Consider now examples of *unbounded* symbols illustrating the above theorems.

EXAMPLE 1. Let
$$a(r) = e^{ir^{2\alpha}}\, r^{2\beta}, \qquad \alpha > 1, \quad \beta > 0.$$
Consider the first average of this symbol
$$\begin{aligned} B_{(1)}(r) &= \int_r^\infty e^{iu^\alpha}\, u^\beta\, e^{r-u} du = \frac{1}{i\alpha}\int_r^\infty u^{\beta-\alpha+1}\, e^{r-u} de^{iu^\alpha} \\ &= -\frac{e^{ir^\alpha} r^{\beta-\alpha+1}}{i\alpha} - \frac{1}{i\alpha}\int_r^\infty e^{iu^\alpha}[(\beta-\alpha+1)u^{\beta-\alpha} - u^{\beta-\alpha+1}]e^{r-u}du. \end{aligned}$$
If
$$\beta - \alpha + 1 < 0, \tag{4.2}$$
then the function $B_{(1)}(r)$ tends to zero when $r \to \infty$, and the operator T_a is compact. If the condition (4.2) does not hold, then integrating by parts several times we arrive to
$$B_{(1)}(r) = \sum_{k=1}^m c_k\, e^{ir^\alpha} r^{\beta_k} + \int_r^\infty e^{iu^\alpha}\left(\sum_{k=1}^{m_1} d_k r^{\xi_k}\right) e^{r-u} du, \tag{4.3}$$
where β_k is a decreasing sequence of real numbers, since $1 - \alpha < 0$. Integrating by parts as much as necessary we obtain that all numbers ξ_k are non positive. Thus the integral summand in (4.3) is a bounded function. In fact, formula (4.3) gives an asymptotic representation of $B_{(1)}(r)$ when $r \to \infty$ with the principal term having the form
$$B^o_{(1)}(r) \sim e_1 e^{ir^\alpha} r^{\beta-\alpha+1}.$$
Consider next averages $B_{(j)}(r)$ of our symbol. Repeating the above calculation we obtain that the principal terms of the asymptotic expansion of the functions $B_{(j)}(r)$ near the infinity have the form
$$B^o_{(j)}(r) \sim e_j e^{ir^\alpha} r^{\beta-j(\alpha+1)}.$$
For sufficiently large j the function $B_{(j)}(r)$ tends to zero at infinity, and thus by Theorem 4.3 the operator T_a is compact.

It is worth mentioning that not all symbols can be treated by Theorem 4.3 by passing to an appropriate sufficiently large average. The next example illustrates this phenomena.

EXAMPLE 2. Let now
$$a(r) = e^{ir^2} r^{2m}, \qquad m \in \mathbb{N},$$
that is, comparing with the previous example, $\alpha = 1$, $\beta = m \in \mathbb{N}$.

Let us try to apply Theorem 4.3 to this symbol. For the first average we have
$$B_{(1)}(r) = \int_r^\infty e^{iu} u^m e^{r-u} du$$

$$
\begin{aligned}
&= \frac{e^r}{i-1}\int_r^\infty u^m de^{(i-1)u} \\
&= \frac{e^{ir}r^m}{1-i} - \frac{m}{i-1}\int_r^\infty e^{iu}u^{m-1}e^{r-u}du.
\end{aligned}
$$

Now we see that the principal term of asymptotics (up to a multiplicative constant) has the same form as the initial function $b(r) = a(\sqrt{r})$:

$$B_{(1)}(r) \sim \frac{1}{1-i}\cdot e^{ir}r^m.$$

Analogously,

$$B_{(j)}(r) \sim \left(\frac{1}{1-i}\right)^j \cdot e^{ir}r^m.$$

Thus Theorem 4.3 is not applicable to this symbol.

At the same time:

$$\gamma_a(n) = \frac{1}{n!}\int_0^\infty e^{ir}e^{-r}r^{n+m}dr.$$

Integrating by parts $(n+m)$ times we have

$$\gamma_a(n) = \frac{(n+m)!}{(i-1)^{n+m}n!}.$$

Estimate the modulus

$$|\gamma_a(n)| \leq \text{const}\,\frac{n^m}{2^{\frac{n+m}{2}}}.$$

Thus the sequence $\{\gamma_a(n)\}$ tends to zero as n tends to infinity, and the Toeplitz operator T_a is compact, while all the averages $B_{(j)}(r)$ are unbounded.

Example 3. Let $\varphi(r)$ be an infinitely differentiable function on $\mathbb{R}_+$ monotonically increasing to infinity together with its own first derivative as r tends to infinity. Consider the following symbol

$$a(r) = e^{i\varphi(r^2)}\varphi'(r^2).$$

Then

$$
\begin{aligned}
B_{(1)}(r) &= \int_r^\infty e^{i\varphi(u)}\varphi'(u)e^{r-u}du \\
&= ie^{i\varphi(r)} + i\int_r^\infty e^{i\varphi(u)}e^{r-u}du,
\end{aligned}
$$

and

$$\left|\int_r^\infty e^{i\varphi(u)}e^{r-u}du\right| \leq \int_r^\infty e^{r-u}du = 1.$$

Thus the function $B_{(1)}(r)$ is bounded. It can be shown analogously that the principal asymptotic term of the second average $B_{(2)}(r)$ has the form

$$B_{(2)}(r) \sim \frac{e^{i\varphi(r)}}{\varphi'(r)}.$$

Thus by Theorem 4.3 the Toeplitz operator T_a is compact.

The function $\varphi(r)$ can be taken, for example, in the form

$$\varphi(r) = e^{r^d}, \qquad 0 < d < 1.$$

In this case the modulus $|a(r)|$ is a very rapidly increasing function. It increases more rapidly that any power function, nevertheless the Toeplitz operator T_a is still compact.

Example 4. Consider now an example of a non negative unbounded symbol $a(r)$ for which the corresponding Toeplitz operator T_a is compact. Let $a(r)$ be the following piecewise constant function

$$a(\sqrt{r}) = \begin{cases} n, & r \in [n, n+n^{-3}] \\ 0, & r \in (n+n^{-3}, n+1) \end{cases} .$$

Estimate the first average

$$\begin{aligned} |B_{(1)}(r)| &= \int_r^\infty a(\sqrt{u})e^{r-u}du \\ &\leq \int_{[r]}^\infty a(\sqrt{u})\,du = \sum_{m=[r]}^\infty \frac{1}{m^2} \leq \text{const}\,\frac{1}{[r]}, \end{aligned}$$

where $[r]$ denotes the integer part of r. That is

$$\lim_{r\to\infty} B_{(1)}(r) = 0,$$

and the operator T_a is compact.

Consider now the spectral properties of bounded Toeplitz operators T_a with symbols $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$. First of all, as a direct corollary of Corollary 3.3 and Theorem 3.7 we have

Theorem 4.4 *For any compact set $M \in \mathbb{C}$ there exists a bounded Toeplitz operator T_a with symbol $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ such that*

$$\operatorname{sp} T_a = \text{ess-sp}\, T_a = M.$$

In particular, from this theorem it follows that the essential spectrum of a Toeplitz operator is not always connected. At the same time in the theory of Toeplitz operators the connectedness of the essential spectrum theorems play an important role. Thus it interesting to find sufficient conditions which guarantee the connectedness of the essential spectrum.

Theorem 4.5 *Let* $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$. *If*

$$\lim_{n\to\infty} (\gamma_a(n+1) - \gamma_a(n)) = 0, \tag{4.4}$$

then the essential spectrum of the Toeplitz operator T_a *is connected.*

Proof. See the proof of Corollary 3.8 in [13]. $\square$

Let us express now the condition (4.4) in the terms of a symbol $a(r)$ directly. Introduce

$$\gamma_a^{(1)}(n) = \gamma_a(n+1) - \gamma_a(n).$$

Integrating by parts the expression for $\gamma_a(n+1)$ we have

$$\begin{aligned}
\gamma_a^{(1)}(n) &= \frac{1}{n!}\int_0^\infty \left(\int_r^\infty a(\sqrt{u})\, e^{r-u} du\right) e^{-r} r^n dr - \frac{1}{n!}\int_0^\infty a(\sqrt{r})\, e^{-r} r^n dr \\
&= \frac{1}{n!}\int_0^\infty a_1(\sqrt{r})\, e^{-r} r^n dr,
\end{aligned}$$

where

$$a_1(\sqrt{r}) = \int_r^\infty (a(\sqrt{u}) - a(\sqrt{r})) e^{r-u} dr. \tag{4.5}$$

Thus we have immediately

Corollary 4.6 *Given* $a(r) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$, *if the Toeplitz operator* T_{a_1} *with the symbol* $a_1(r)$ *of the form (4.5) is compact, then the essential spectrum of the Toeplitz operator* T_a *with the initial symbol* $a(r)$ *is connected.*

In particular, applying Theorem 4.3 we have

Corollary 4.7 *Let the function* $a(r)$ *be differentiable, and the derivative*

$$\frac{da(\sqrt{r})}{dr} = \frac{1}{2\sqrt{r}}\, a'(\sqrt{r})$$

tend to zero as r *tends to infinity. Then the essential spectrum of the Toeplitz operator* T_a *is connected.*

5 Algebras of Toeplitz and Weyl pseudodifferential operators

Recall first the essential ingredients of the Berezin theory (see, for example, [4, 5, 14]).

A Toeplitz operator T_a with a symbol $a = a(z)$ acting on the Fock space $F^2(\mathbb{C})$ from this point of view is an operator with *anti-Wick symbol* $a = a(z)$. The function $\widetilde{a}(z, \overline{z})$ is called a *Wick symbol* of an operator T if this operator acts on $F^2(\mathbb{C})$ as follows

$$\begin{aligned}
(Tf)(z) &= \frac{1}{\pi}\int_{\mathbb{C}} \widetilde{a}(z, \overline{\zeta}) f(\zeta) e^{-\overline{\zeta}(\zeta - z)} dv(\zeta) \\
&= \int_{\mathbb{C}} \widetilde{a}(z, \overline{\zeta}) f(\zeta) e^{\overline{\zeta} z} d\mu(\zeta).
\end{aligned} \tag{5.1}$$

The Wick and anti-Wick symbols of the same operator are connected by the formula

$$\widetilde{a}(z,\overline{z}) = \frac{1}{\pi}\int_{\mathbb{C}} e^{-(z-\zeta)(\overline{z}-\overline{\zeta})}\, a(\zeta)\, dv(\zeta).$$

Let T_1 and T_2 be two operators with Wick symbols $\widetilde{a}_1$ and $\widetilde{a}_2$ respectively, then for the Wick symbol $\widetilde{a}$ of the operator $T = T_1T_2$ the following composition formula holds:

$$\widetilde{a}(z,\overline{z}) = (\widetilde{a}_1 \star \widetilde{a}_2)(z,\overline{z}) = \frac{1}{\pi}\int_{\mathbb{C}} \widetilde{a}_1(z,\overline{\zeta})\, \widetilde{a}_2(\zeta,\overline{z})\, e^{-(z-\zeta)(\overline{z}-\overline{\zeta})}\, dv(\zeta). \tag{5.2}$$

The Bargmann transform ([2]) $B : L_2(\mathbb{R}) \longrightarrow F^2(\mathbb{C})$, where

$$(B\psi)(z) = \pi^{-\frac{1}{4}}\int_{\mathbb{R}} e^{-\frac{1}{2}(x^2-2\sqrt{2}xz+z^2)}\psi(x)\, dx, \tag{5.3}$$

is an isometrical isomorphism, and the inverse isomorphism $B^{-1} = B^* : F^2(\mathbb{C}) \longrightarrow L_2(\mathbb{R})$ is given by

$$(B^{-1}f)(x) = \pi^{-\frac{1}{4}}\int_{\mathbb{C}} e^{-\frac{1}{2}(x^2-2\sqrt{2}x\overline{z}+\overline{z}^2)} f(z)\, d\mu(z).$$

Now each Toeplitz operator T_a with the (anti-Wick) symbol $a = a(z)$ acting on the Fock space $F^2(\mathbb{C})$ is unitary equivalent to the operator $\widehat{T}_a = B^{-1}T_aB$ acting on $L_2(\mathbb{R})$, which is a *Weyl pseudodifferential* operator. We will denote by $a_w(x,\xi)$ its *Weyl symbol.*

The (anti-Wick) symbol $a = a(z)$ and the Weyl symbol $a_w(x,\xi)$ of the operators T_a and $\widehat{T}_a = B^{-1}T_aB$, respectively, are connected by the formula

$$a_w(x,-\xi) = \frac{2}{\pi}\int_{\mathbb{C}} a(\zeta)\, e^{-2(z-\zeta)(\overline{z}-\overline{\zeta})} dv(\zeta), \tag{5.4}$$

where $z = \frac{1}{\sqrt{2}}(x+i\xi)$.

Let us mention as well the connection between the Weyl and the Wick symbols of the operators $\widehat{T}_a = B^{-1}T_aB$ and T_a respectively:

$$\widetilde{a}(z,\overline{z}) = \frac{2}{\pi}\int_{\mathbb{C}} a_w(x,-\xi) e^{-2(z-\zeta)(\overline{z}-\overline{\zeta})} dv(\zeta),$$

where $\zeta = \frac{1}{\sqrt{2}}(x+i\xi)$.

We consider now the C^*-algebra generated by bounded Toeplitz operators with radial symbols. First observe that our class of symbols, as well as the corresponding Toeplitz C^*-algebra will have certain peculiarities. In particular, contrary to commonly known and studied cases, the Toeplitz operator algebra is commutative, yet the semicommutators $[T_{a_1}, T_{a_2}) = T_{a_1} \cdot T_{a_2} - T_{a_1 \cdot a_2}$ are not compact in general. Moreover, the symbols under consideration do not form an algebra (under pointwise multiplication). That is, given two radial symbols $a_1(r)$ and $a_2(r)$, for which the corresponding Toeplitz operators $T_{a_1(r)}$ and $T_{a_2(r)}$ are bounded, the Toeplitz operator $T_{a_1 \cdot a_2}$, which corresponds to the product of these symbols, is not necessarily bounded. The natural structure on the set of symbols under consideration is a linear space (in the algebraic sense, i.e., no norm structure assumed).

Example 5. Consider the following radial symbols

$$a_1(r) = e^{ir^{2\alpha}} r^{2\beta} \qquad \text{and} \qquad a_2(r) = e^{-ir^{2\alpha}} r^{2\beta},$$

where $\alpha > 1$ and $\beta > 0$. Then by the results of Example 1 both operators T_{a_1} and T_{a_2} are *bounded*. At the same time the Toeplitz operator T_{a_3} with symbol

$$a_3(r) = a_1(r) \cdot a_2(r) = e^{4\beta}$$

is *unbounded* by the statement 3) of Theorem 4.3.

Denote by $\mathcal{M}$ the linear subspace of $L_1^{\infty}(\mathbb{R}_+, e^{-r^2})$ such that for each $a(r) \in \mathcal{M}$ the Toeplitz operator $T_{a(r)}$ is bounded on $F^2(\mathbb{C})$, and denote by $\mathcal{T}(\mathcal{M})$ the C^*-algebra generated by all Toeplitz operators T_a with symbols $a \in \mathcal{M}$.

Theorem 5.1 *The C^*-algebra $\mathcal{T}(\mathcal{M})$ is commutative, and isomorphically isometric to the algebra l_∞. The isomorphism*

$$\nu : \mathcal{T}(\mathcal{M}) \longrightarrow l_\infty$$

is generated by the following mapping

$$\nu : T_a \longmapsto \gamma_a,$$

where $a(r) \in \mathcal{M}$, and the sequence γ_a is given by (3.2).

Proof. Follows directly from Theorems 3.1, 3.2, and 3.7. □

Remark 5.2 Considering an algebra $\mathcal{T}$ generated by Toeplitz operators T_a with symbols from a certain class one typically has that the elements of the algebra $\mathcal{T}$ have in general more complicated structure than the initial generators T_a. In that sense our algebra $\mathcal{T}(\mathcal{M})$ is quite unusual: *each element of the algebra $\mathcal{T}(\mathcal{M})$ is just a Toeplitz operator*, whose symbol can be recovered from the corresponding l_∞ sequence by the procedure of Theorem 3.7.

The commutativity of our Toeplitz operator algebra $\mathcal{T}(\mathcal{M})$ implies, in particular, that this algebra has a very rich structure of invariant subspaces.

Theorem 5.3 *Let Λ be an arbitrary (finite or infinite) subset of $\mathbb{Z}_+$. Then the subspace*

$$\mathcal{A}_\Lambda = \{\psi(z) = \sum_{n\in\Lambda} c_n z^n \, : \, \psi \in F^2(\mathbb{C})\}$$

is invariant for the algebra $\mathcal{T}(\mathcal{M})$. Moreover the orthogonal projection P_Λ onto the subspace $\mathcal{A}_\Lambda$ belongs to the algebra $\mathcal{T}(\mathcal{M})$, and thus is a Toeplitz operator with symbol from $\mathcal{M}$.

In particular, each subspace of polynomials (with fixed set of powers of their terms) is invariant for the algebra $\mathcal{T}(\mathcal{M})$.

PROOF. Follows directly from Theorems 3.1 and 2.3, and Remark 5.2. $\square$

The system of functions

$$\ell_{(n)}(z) = \frac{z^n}{\sqrt{n!}}, \qquad n \in \mathbb{Z}_+,$$

is an orthonornal base for the Fock space $F^2(\mathbb{C})$. Denote by $L_{(n)}$ the one-dimensional space generated by the function $\ell_{(n)}(z)$. The orthogonal projection $P_{(n)} : F^2(\mathbb{C}) \longrightarrow L_{(n)}$ obviously has the form

$$(P_{(n)}f)(z) = \langle f(\zeta), \ell_{(n)}(\zeta)\rangle \, \ell_{(n)}(z) = \frac{z^n}{n!}\,\frac{1}{\pi}\int_{\mathbb{C}} f(\zeta)\,\overline{\zeta}^n e^{-|\zeta|^2} dv(\zeta), \tag{5.5}$$

and is a Toeplitz operator with symbol from $\mathcal{M}$.

Corollary 5.4 *For any $n \in \mathbb{Z}_+$ the one-dimensional space $L_{(n)}$ is an eigenspace for any Toeplitz operator T_a with $a(r) \in \mathcal{M}$, and the corresponding eigenvalue is equal to $\gamma_a(n)$.*

Theorem 5.5 *Let $a(r) \in \mathcal{M}$. Then the Wick symbol of the Toeplitz operator T_a is radial as well, and is calculated by the formula*

$$\widetilde{a}(z, \overline{z}) = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}\, \gamma_a(n). \tag{5.6}$$

PROOF. The proof is a matter of calculation:

$$\begin{aligned}
\widetilde{a}(z,\overline{z}) = \widetilde{a}(|z|) &= \frac{1}{\pi}\int_{\mathbb{C}} e^{-(z-\zeta)(\overline{z}-\overline{\zeta})}\, a(|\zeta|)\, dv(\zeta) \\
&= \frac{1}{\pi i}\int_{\mathbb{R}_+}\int_{S^1} e^{-(|z|^2+r^2-\overline{z}rt-zrt^{-1})}\,\frac{dt}{t}\, a(r)\, rdr \\
&= \frac{e^{-|z|^2}}{\pi i}\int_{\mathbb{R}_+}\left(\int_{S^1} e^{(\overline{z}r)t} e^{(zr)t^{-1}}\,\frac{dt}{t}\right) a(r) e^{-r^2} rdr \\
&= \frac{e^{-|z|^2}}{\pi i}\int_{\mathbb{R}_+}\int_{S^1}\left(\sum_{n=0}^{\infty}\frac{(\overline{z}r)^n t^n}{n!}\sum_{k=0}^{\infty}\frac{(zr)^n t^{-k}}{k!}\right)\frac{dt}{t}\, a(r)\, e^{-r^2} rdr \\
&= e^{-|z|^2}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}\,\frac{2}{n!}\int_{\mathbb{R}_+} a(r) e^{-r^2} r^{2n+1} dr \\
&= e^{-|z|^2}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}\,\gamma_a(n).
\end{aligned}$$

$\square$

Berger and Coburn stated in [8] a conjecture that for a certain class of symbols Toeplitz operator T_a is bounded if and only if its Wick symbol is bounded. Note, that in general case of radial symbols the boundedness of a Wick symbol does not guarantee the boundedness of the sequence $\{\gamma_a(n)\}$, which is equivalent to boundedness of Toeplitz operator with radial symbol a. Indeed, following all the steps of the proof of Theorem 3.7 one can construct a symbol $a = a(r)$ such that the corresponding sequence γ_a is given by

$$\gamma_a(n) = (-1)^{n-1} n.$$

For such a symbol the Toeplitz operator is obviously unbounded, while its Wick symbol

$$\widetilde{a}(z, \overline{z}) = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} (-1)^{n-1} n = |z|^2 \, e^{-2|z|^2}$$

is not only bounded, but even tends to 0 when $z \to \infty$. That is, the Berger-Coburn conjecture is not true in general.

Moreover, this example, as well as the sequence

$$\gamma_a(n) = (-1)^n,$$

which is generated by a *bounded* Toeplitz operator, shows that in the Fock space setting the compactness of Toeplitz operator *is not equivalent* to the vanishing of its Wick symbol when $z \to \infty$. In this context recall that in the Bergman space setting [1] the compactness of a Toeplitz operator *is equivalent* to the vanishing of its Wick symbol when $z \to \partial D$.

Corollary 5.6 *Let $a(r) \in \mathcal{M}$. Writing the Toeplitz operator T_a in the form of an operator with Wick symbol (5.1) gives the spectral decomposition of the operator T_a:*

$$T_a = \sum_{n=0}^{\infty} \gamma_a(n) P_{(n)} \tag{5.7}$$

PROOF. Given $n \in \mathbb{Z}_+$, consider the operator with the Wick symbol of the form $\widetilde{p}_{(n)}(z, \overline{z}) = e^{-z\overline{z}} \frac{z^n \overline{z}^n}{n!}$

$$\frac{1}{\pi} \int_{\mathbb{C}} e^{-z\overline{\zeta}} \frac{z^n \overline{\zeta}^n}{n!} e^{-\overline{\zeta}(\zeta - z)} f(\zeta) \, dv(\zeta) = \frac{z^n}{n!} \frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \, \overline{\zeta}^n e^{-|\zeta|^2} dv(\zeta) = (P_{(n)} f)(z),$$

where $P_{(n)}$ is the orthogonal projection (5.5). Thus the Wick symbol (5.6) of the operator T_a admits the representation

$$\widetilde{a}(z, \overline{z}) = \sum_{n=0}^{\infty} \gamma_a(n) \widetilde{p}_{(n)}(z, \overline{z}),$$

which proves the theorem. □

Remark 5.7 In addition to Theorem 4.4, formula (5.7) gives an alternative way to construct a Toeplitz operator with *predefined* spectrum, essential spectrum, or eigenvalues corresponding to eigenspaces $L_{(n)}$, $n \in \mathbb{Z}_+$.

The set $W(\mathcal{M})$ of all Wick symbols for Toeplitz operators T_a with (anti-Wick) symbols $a(r) \in \mathcal{M}$ is obviously coincides with the set of all functions of the form (5.6)

$$\widetilde{a}(z,\overline{z}) = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}\, \gamma(n),$$

where $\gamma = \{\gamma(n)\}_{n\in\mathbb{Z}_+} \in l_\infty$. This set is obviously a linear space, and the multiplication law (5.2) in our case has the form: let

$$\widetilde{a}_1(z,\overline{z}) = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}\, \gamma_1(n) \qquad \text{and} \qquad \widetilde{a}_1(z,\overline{z}) = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}\, \gamma_2(n)$$

then

$$\widetilde{a}(z,\overline{z}) = (\widetilde{a}_1 \star \widetilde{a}_2)(z,\overline{z}) = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}\, \gamma_1(n)\, \gamma_2(n). \tag{5.8}$$

This can be seen either from (5.7), or by direct computation:

$$\begin{aligned}
\widetilde{a}(z,\overline{z}) &= (\widetilde{a}_1 \star \widetilde{a}_2)(z,\overline{z}) = \frac{1}{\pi}\int_{\mathbb{C}} \widetilde{a}_1(z,\overline{\zeta})\, \widetilde{a}_2(\zeta,\overline{z})\, e^{-(z-\zeta)(\overline{z}-\overline{\zeta})}\, dv(\zeta) \\
&= \frac{1}{\pi}\int_{\mathbb{C}} e^{-z\overline{\zeta}} e^{-\zeta\overline{z}} \left(\sum_{n=0}^{\infty} \frac{(z\overline{\zeta})^n}{n!} \gamma_1(n) \sum_{k=0}^{\infty} \frac{(\zeta\overline{z})^k}{k!} \gamma_2(k) \right) e^{-(|z|^2+|\zeta|^2} e^{z\overline{\zeta}} e^{\zeta\overline{z}} dv(\zeta) \\
&= e^{-|z|^2} \frac{1}{\pi i} \int_{\mathbb{R}_+} \left(\int_{S^1} \sum_{n=0}^{\infty} \frac{(zr)^n t^{-n}}{n!} \gamma_1(n) \sum_{k=0}^{\infty} \frac{(\overline{z}r)^k t^k}{k!} \gamma_2(k) \frac{dt}{t} \right) e^{-r^2} r dr \\
&= e^{-|z|^2} 2 \int_{\mathbb{R}_+} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(n!)^2}\, \gamma_1(n)\, \gamma_2(n)\, e^{-r^2} r^{2n+1} dr \\
&= e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}\, \gamma_1(n)\, \gamma_2(n).
\end{aligned}$$

That is, the linear space $W(\mathcal{M})$ is a commutative algebra with respect to multiplication (5.8), and the Toeplitz operator algebra $\mathcal{T}(\mathcal{M})$ (besides the isomorphism of Theorem 5.1) is isomorphic to $W(\mathcal{M})$ via the following mapping

$$\omega:\ T_a \in \mathcal{T}(\mathcal{M}) \longmapsto \widetilde{a}(z,\overline{z}) = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}\, \gamma_a(n) \in W(\mathcal{M}).$$

Under the Bargmann transform (5.3) the algebra $\mathcal{T}(\mathcal{M})$ is unitary equivalent to a certain algebra of Weyl pseudodifferential operators:

$$\Psi(\mathcal{M}) = B^{-1}\mathcal{T}(\mathcal{M})B = \{\mathrm{Op}\,(a_w) = \widehat{T}_a = B^{-1}T_aB :\ a = a(r) \in \mathcal{M}\}.$$

Theorem 5.8 *Let $a(r) \in \mathcal{M}$. Then the Weyl symbol of the operator* $\mathrm{Op}\,(a_w) = B^{-1}T_aB$ *is radial as well, and is calculated by the formula*

$$a_w(x,\xi) = e^{-(x^2+\xi^2)} \sum_{n=0}^{\infty} \frac{(x^2+\xi^2)^n}{n!} \, \gamma_{a(\frac{r}{\sqrt{2}})}(n). \tag{5.9}$$

Proof. Direct calculations using formula (5.4). $\square$

Recall that the function

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2x)^{n-2m}}{m!\,(n-2m)!}$$

is the Hermite polinomial of degree n (see, for example [3, 20]), and that the functions

$$h_n(x) = (2^n n! \sqrt{\pi})^{-1/2} \, H_n(x) \, e^{-x^2/2}, \qquad n \in \mathbb{Z}_+$$

form an orthonormal base in $L_2(\mathbb{R})$. Denote by H_n the one-dimensional subspace of $L_2(\mathbb{R})$ generated by the function $h_n(y)$, and by Q_n the one-dimensional orthogonal projection of $L_2(\mathbb{R})$ onto H_n, which is given obviously by

$$(Q_n\psi)(y) = h_n(x) \int_{\mathbb{R}} \psi(\eta) \, h_n(\eta) \, d\eta.$$

Reformulate now the above statements about Toeplitz operators for the our class of Weyl operators.

Theorem 5.9 *The C^*-algebra $\Psi(\mathcal{M})$ is commutative, and isomorphically isometric to the algebra l_∞. The isomorphism*

$$\omega : \Psi(\mathcal{M}) \longrightarrow l_\infty$$

is generated by the following mapping

$$\omega : \mathrm{Op}\,(a_w) = B^{-1}T_aB \longmapsto \gamma_a,$$

where $a(r) \in \mathcal{M}$, $a_w(x,\xi)$ is given by (5.9), and the sequence γ_a is given by (3.2).

For any $n \in \mathbb{Z}_+$ the one-dimensional space H_n is an eigenspace for any operator $\mathrm{Op}\,(a_w) \in \Psi(\mathcal{M})$*, and the corresponding eigenvalue is equal to $\gamma_a(n)$.*

The Wick symbol $\widetilde{a}(|z|)$ of an operator $\mathrm{Op}\,(a_w) \in \Psi(\mathcal{M})$ *is given by (5.6).*

Any operator $\mathrm{Op}\,(a_w) \in \Psi(\mathcal{M})$ *admits the following spectral decomposition*

$$\mathrm{Op}\,(a_w) = \sum_{n=0}^{\infty} \gamma_a(n) Q_n,$$

where, besides the formula (3.2), the eigenvalues $\gamma_a(n)$ can be calculated by the formula

$$\begin{aligned} \gamma_a(n) &= \frac{d^n}{dr^n} \left(e^{r^2} \widetilde{a}(\sqrt{r}) \right)_{r=0} \\ &= \frac{d^n}{dr^n} \left(\frac{1}{\pi} \int_{\mathbb{R}^2} a_w(x,\xi) \, e^{-|(x+i\xi)-\sqrt{2r}|^2 + r^2} dx d\xi \right)_{r=0}. \end{aligned}$$

There exists an operator $\mathrm{Op}\,(a_w) \in \Psi(\mathcal{M})$ *with any* predefined *(compact) spectrum, essential spectrum, or a bounded sequence of eigenvalues corresponding to eigenspaces* H_n, $n \in \mathbb{Z}_+$.

It is evident that most results of this section remain valid (with appropriate changes and usual care about domains) for unbounded Toeplitz operators with radial symbols from $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, and for the corresponding unbounded Weyl pseudodifferential operators.

Example 4. Harmonic oscillator.
In the space $L_2(\mathbb{R})$ introduce the creation and annihilation operators

$$a^\dagger = \frac{1}{\sqrt{2}}(x - \frac{d}{dx}), \qquad a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx}),$$

and consider the harmonic oscillator

$$H = \frac{1}{2}(a^\dagger a + aa^\dagger) = \frac{1}{2}(x^2 - \frac{d^2}{dx^2}). \tag{5.10}$$

Passing to the Fock space we have obviously

$$\mathbf{a}^\dagger = B\, a^\dagger B^{-1} = z, \qquad \mathbf{a} = B\, aB^{-1} = \frac{d}{dz},$$

and

$$\mathbf{H} = B\, HB^{-1} = \frac{1}{2}(\mathbf{a}^\dagger \mathbf{a} + \mathbf{a}\mathbf{a}^\dagger) = \frac{1}{2}(z\frac{d}{dz} + \frac{d}{dz}z).$$

The operator $\mathbf{H}$ acts on elements of the base in $F^2(\mathbb{C})$ as follows

$$\mathbf{H}\frac{z^n}{\sqrt{n!}} = \frac{2n+1}{2}\frac{z^n}{\sqrt{n!}}, \qquad n \in \mathbb{Z}_+,$$

and thus it coincides with the Toeplitz operator T_h with the symbol $h(r) = r^2 - \frac{1}{2}$, which has the same sequence of eigenvalues

$$\begin{aligned}
\gamma_h(n) &= \frac{1}{n!}\int_{\mathbb{C}} h(\sqrt{r})e^{-r}r^n dr \\
&= \frac{1}{n!}\int_{\mathbb{C}} e^{-r}r^{n+1}dr - \frac{1}{2} = (n+1) - \frac{1}{2} = \frac{2n+1}{2}.
\end{aligned}$$

Now, $h(\frac{r}{\sqrt{2}}) = \frac{1}{2}(r^2 - 1)$, and

$$\gamma_{h(\frac{r}{\sqrt{2}})}(n) = \frac{1}{2}\left(\frac{1}{n!}\int_{\mathbb{C}} e^{-r}r^{n+1}dr - 1\right) = \frac{n}{2}.$$

Calculating by (5.9) the Weyl symbol of the harmonic oscillator (5.10) we have

$$\begin{aligned}
h_w(x,\xi) &= e^{-(x^2+\xi^2)}\sum_{n=0}^{\infty}\frac{(x^2+\xi^2)^n}{n!}\,\frac{n}{2} \\
&= \frac{x^2+\xi^2}{2}\,e^{-(x^2+\xi^2)}\sum_{n=1}^{\infty}\frac{(x^2+\xi^2)^{n-1}}{(n-1)!} = \frac{1}{2}(x^2+\xi^2).
\end{aligned}$$

In this connection note that both the Toeplitz and Weyl pseudodifferential operators we are considering are nothing but functions of the harmonic oscillator, considered in the Fock space $F^2(\mathbb{C})$ or in $L_2(\mathbb{R})$, respectively.

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