ScienceDirect

Linear Algebra and its Applications
Volume 653, 15 November 2022, Pages 86-115

Eigenvalues of the laplacian matrices of the cycles with one weighted edge

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https://doi.org/10.1016/j.laa.2022.07.011

Abstract

In this paper we study the eigenvalues of the laplacian matrices of the cyclic graphs with one edge of weight α and the others of weight 1. We denote by n the order of the graph and suppose that n tends to infinity. We notice that the characteristic polynomial and the eigenvalues depend only on $\mathrm{Re}(\alpha)$. After that, through the rest of the paper we suppose that $0 < \alpha < 1$. It is easy to see that the eigenvalues belong to $[0, 4]$ and are asymptotically distributed as the function $g(x) = 4\sin^2(x/2)$ on $[0, \pi]$. We obtain a series of results about the individual behavior of the eigenvalues. First, we describe more precisely their localization in subintervals of $[0, 4]$. Second, we transform the characteristic equation to a form convenient to solve by numerical methods. In particular, we prove that Newton's method converges for every $n \geq 3$. Third, we derive asymptotic formulas for all eigenvalues, where the errors are uniformly bounded with respect to the number of the eigenvalue.

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MSC

05C50; 15B05; 47B36; 15A18; 41A60; 65F15; 82B20

Keywords

Eigenvalue; Laplacian matrix; Weighted cycle; Toeplitz matrix; Perturbation; Asymptotic expansion

1. Introduction

For every natural $n \geq 3$ and every real α, we denote by $G_{\alpha,n}$ the cyclic graph of order n, where the edge between the vertices 1 and n has weight α, and all other edges have weights 1. See Fig. 1 for $n = 7$.

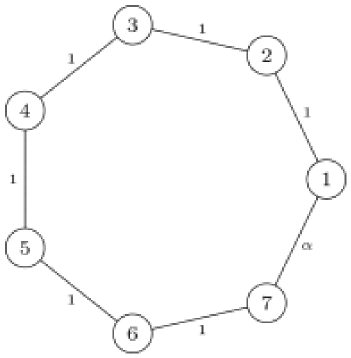

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Fig. 1. Graph $G_{\alpha,7}$.

Let $L_{\alpha,n}$ be the laplacian matrix of $G_{\alpha,n}$. For example,

$$L_{\alpha,7} = \begin{bmatrix} 1+\alpha & -1 & 0 & 0 & 0 & 0 & -\alpha \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -\alpha & 0 & 0 & 0 & 0 & -1 & 1+\alpha \end{bmatrix}. \tag{1}$$

The spectral decomposition of $L_{\alpha,n}$ is crucial to solve the heat and wave equations on the graph $G_{\alpha,n}$, i.e., the linear systems of differential equations of the form $f'(t) = -cL_{\alpha,n}f(t)$ and $f''(t) = cL_{\alpha,n}f(t)$, where $f(t) = [f_j(t)]_{j=1}^n$ and c is some coefficient. Moreover, laplacian matrices appear in the study of random walks on graphs, electrical flows, network dynamics, and many other physical phenomena; see, e.g. [21]. There are plenty of investigations about the spectral properties of laplacian matrices; for example, [1], [14] deal with some families of laplacian matrices having locally Toeplitz structure.

The matrices $L_{\alpha,n}$ can also be viewed as periodic Jacobi matrices and as real symmetric Toeplitz matrices with perturbations on the corners $(1,1)$, $(1,n)$, $(n,1)$, and (n,n). The eigenvalues are explicitly known only for some very special matrix families from these classes; mainly when the eigenvectors are the columns of the DCT or DST matrices [9].

Over the past decade, there has been an increasing interest in Toeplitz matrices with certain perturbations, see [4], [8], [9], [12], [13], [16], [23], [24], [29], [30], [34], or [19], [22], [25], [31] for more general researches. In [12], [13], the characteristic polynomials were computed for some cases of Toeplitz matrices with corner perturbations.

The asymptotic distribution of the eigenvalues of hermitian Toeplitz matrices with small-rank perturbations is described by analogs of Szegő theorem [15], [27], [28]. The individual behavior of the eigenvalues is known only for some particular cases, including hermitian Toeplitz matrices with simple-loop symbols [3], [5], [6], [7].

The methods used in the present paper are similar to the ones from [17], where we studied the eigenvalues of the hermitian tridiagonal Toeplitz matrices with diagonals $-1, 2, -1$ and perturbations $-\alpha$ and $-\overline{\alpha}$ on the corners $(n, 1)$ and $(1, n)$, respectively. In the present paper, we put $1+\alpha$ instead of 2 in the entries $(1, 1)$ and (n, n).

The matrices $L_{\alpha,n}$ are real and symmetric, thus their eigenvalues are real. We enumerate them in the ascending order:

$$\lambda_{\alpha,n,1} \le \lambda_{\alpha,n,2} \le \cdots \le \lambda_{\alpha,n,n}. \tag{2}$$

It is well known that every laplacian matrix has eigenvalue 0 associated to the eigenvector $[1, \ldots, 1]^\top$.

For $\alpha = 0$, the eigenvalues of $L_{0,n}$ are $\lambda_{0,n,j} = g((j-1)\pi/n)$, where g is defined by

$$g(x) := 2 - 2\cos(x) = 4\sin^2 \tfrac{x}{2}, \quad x \in [0, \pi]. \tag{3}$$

The normalized eigenvectors of $L_{0,n}$ are the columns of the matrix DCT-II, see [9, formula (2.53) and (2.54)].

For $\alpha = 1$, the matrices $L_{1,n}$ are circulant, and their eigenvalues and eigenvectors are well known, see, e.g. [17].

It is also well known that the eigenvalues of tridiagonal real symmetric Toeplitz matrices $T_n(g)$ generated by g are $g(j\pi/(n+1))$.

Except for the cases $\alpha = 0$, $\alpha = 1$, and $\alpha = 1/2$ (see Remark 19), we do not know explicit formulas for all eigenvalues of $L_{\alpha,n}$.

For $\alpha < 0$ (resp., $\alpha > 1$), it can be shown that the first (resp., last) eigenvalue goes out the interval $[0, 4]$ and tends exponentially fast to $4\alpha^2/(2\alpha - 1)$. We are going to present the corresponding results in another paper.

In this paper we suppose that $0 < \alpha < 1$.

Our matrices $L_{\alpha,n}$ can be obtained by small-rank perturbations from $T_n(g)$, $L_{0,n}$ or $L_{1,n}$. The Cauchy interlacing theorem or the theory of locally Toeplitz sequences [15], [27], [28] easily implies that the eigenvalues of $L_{\alpha,n}$ are asymptotically distributed as the values of g on $[0, \pi]$, as n tends to infinity.

We obtain much more precise results about the eigenvalues of $L_{\alpha,n}$. Namely, we find exact eigenvalues of the form $g((j-1)\pi/n)$, with j odd, and localize the other eigenvalues in the intervals of the form $(g((j-1)\pi/n), g(j\pi/n))$ with j even.

We transform the characteristic equation to the form $x = f_{\alpha,n,j}(x)$, where $f_{\alpha,n,j}$ is “slow”, i.e., the derivative of $f_{\alpha,n,j}$ is small when n is large. After that, this equation is convenient to solve by the fixed point method and Newton's method (also known as Newton–Raphson or gradient method).

On this base, we derive asymptotic formulas for all eigenvalues $\lambda_{\alpha,n,j}$, where the errors are uniformly bounded on j.

For α in $\mathbb{C}$, we consider the $n \times n$ complex laplacian matrix $L_{\alpha,n}$, for example,

$$L_{\alpha,7} = \begin{bmatrix} 1+\overline{\alpha} & -1 & 0 & 0 & 0 & 0 & -\overline{\alpha} \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -\alpha & 0 & 0 & 0 & 0 & -1 & 1+\alpha \end{bmatrix}. \qquad (4)$$

These matrices appear in the study of problems related to networked multi-agent systems, see [20] for investigations in this area. In Proposition 13 we prove that the characteristic polynomial of $L_{\alpha,n}$ only depends on $\mathrm{Re}(\alpha)$, i.e., $\det(\lambda I_n - L_{\alpha,n}) = \det(\lambda I_n - L_{\mathrm{Re}(\alpha),n})$.

We present the main results of this paper in Section 2, the correspondent proofs lie in Section 4 (localization), Section 5 (main equation), Section 6 (fixed point method), Sections 7 and 8 (Newton's method), Section 9 (asymptotic formulas), Section 10 (norms of the eigenvectors). In Section 3 we give formulas for the characteristic polynomial and eigenvectors of general tridiagonal symmetric Toeplitz matrices with perturbations in the corners $(1,1)$, $(1,n)$, $(n,1)$ and (n,n); our formulas are equivalent to Yueh and Cheng [32]. In Section 11 we show the results of some numerical tests.

As it was commented by the referee, this paper is an example of "exhaustive treatise on a relatively narrow topic". We hope that it may serve as a model case for more general investigations.

2. Main results

We treat α as a fixed parameter, supposing that $0 < \alpha < 1$.

It is well known that 0 is the least eigenvalue of $L_{\alpha,n}$. A direct application of the Gershgorin disks theorem [18, Theorem 6.1.1] shows that all eigenvalues of $L_{\alpha,n}$ belong to $[0,4]$. However, we give a more precise localization.

Theorem 1 eigenvalues' localization

For every $n \geq 3$,

$$\lambda_{\alpha,n,j} = g\left(\tfrac{(j-1)\pi}{n}\right) \quad (j \text{ odd},\ 1 \leq j \leq n), \qquad (5)$$

$$g\left(\tfrac{(j-1)\pi}{n}\right) < \lambda_{\alpha,n,j} < g\left(\tfrac{j\pi}{n}\right) \quad (j \text{ even},\ 1 \leq j \leq n). \qquad (6)$$

In particular, Theorem 1 implies that $\lambda_{\alpha,n,j}$ with odd j does not depend on α.

Motivated by Theorem 1, we use g as a change of variable in the characteristic equation and put

$$d_{n,j} := \tfrac{(j-1)\pi}{n}, \qquad \vartheta_{\alpha,n,j} := \tilde{g}^{-1}(\lambda_{\alpha,n,j}),$$

where $\tilde{g} : [0,\pi] \to [0,4]$ is a restriction of g. In other words, the numbers $\vartheta_{\alpha,n,j}$ belong to $[0,\pi]$ and satisfy $g(\vartheta_{\alpha,n,j}) = \lambda_{\alpha,n,j}$. Then (5) and (6) are equivalent to

$$\vartheta_{\alpha,n,j} = d_{n,j} \quad (j \text{ odd},\ 1 \leq j \leq n),$$
$$d_{n,j} < \vartheta_{\alpha,n,j} < d_{n,j+1} \quad (j \text{ even},\ 1 \leq j \leq n).$$

We define $\eta_\alpha : [0,\pi] \to \mathbb{R}$ by

$$\eta_\alpha(x) := 2\arctan\left(\varkappa_\alpha \cot \tfrac{x}{2}\right), \qquad (7)$$

where

$$\varkappa_\alpha := \frac{\alpha}{1-\alpha}. \quad (8)$$

Obviously, η_α strictly decreases taking values from π to 0. Furthermore, η_α is strictly convex when $0 < \alpha < 1/2$ and strictly concave if $1/2 < \alpha < 1$. Other equivalent formulas for η_α are given in (38), (39), and (40). A direct computation shows that η_α is an involution of the segment $[0, \pi]$, i.e., $\eta_\alpha(\eta_\alpha(x)) = x$ for every x in $[0, \pi]$. This property is not used in the paper. See [33] for the general description of the continuous involutions of real intervals.

Theorem 2 main equation

Let $n \geq 3$ and j be even, $1 \leq j \leq n$. Then the number $\vartheta_{\alpha,n,j}$ is the unique solution of the following equation on $[0, \pi]$:

$$x = d_{n,j} + \frac{\eta_\alpha(x)}{n}. \quad (9)$$

Fig. 2 shows the left-hand side and the right-hand side of (9).

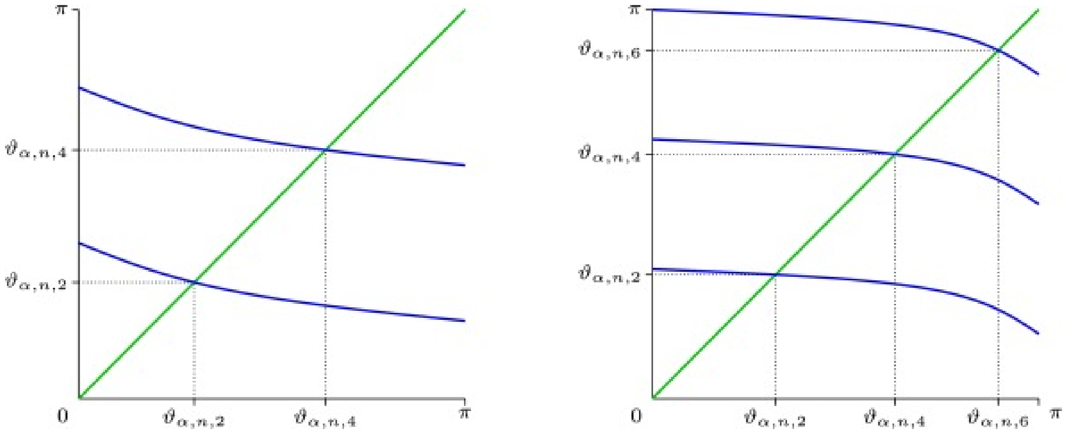

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Fig. 2. The left picture shows the left-hand side (green) and the right-hand side (blue) of (9) for α=1/3, n=5, j=2,4. The right picture corresponds to α=4/5, n=6, j=2,4,6. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

The main equation can be rewritten in the form $nx - (j-1)\pi = \eta_\alpha(x)$. Fig. 3 shows both sides of this equation for some values of α and n, j.

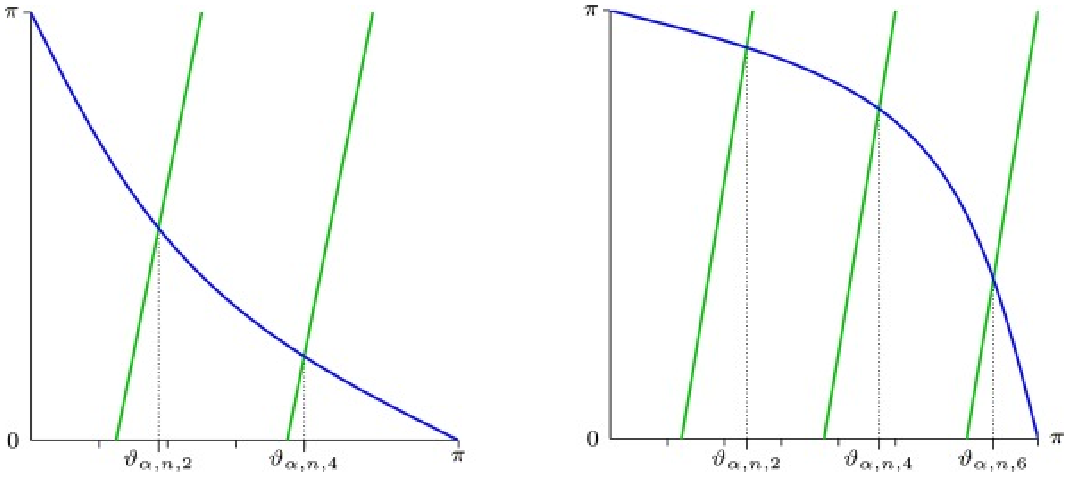

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Fig. 3. Plots of $x \mapsto nx-(j-1)\pi$ (green) and η_α (blue), for $\alpha=1/3$, $n=5$ (left) and $\alpha=4/5$, $n=6$ (right).

For every j with $1 \le j \le n$, we define $I_{n,j} := \left(\frac{(j-1)\pi}{n}, \frac{j\pi}{n}\right)$.

For every $n \ge 3$ and every j even with $1 \le j \le n$, we define $h_{\alpha,n,j} : \mathrm{cl}(I_{n,j}) \to \mathbb{R}$ by

$$h_{\alpha,n,j}(x) := nx - (j-1)\pi - \eta_\alpha(x). \tag{10}$$

In Proposition 18 we show that $h_{\alpha,n,j}$ changes its sign in $I_{n,j}$. Hence, it is feasible to solve (9) by the bisection method or false rule method.

In Proposition 20 we study the dependence of $\lambda_{\alpha,n,j}$ on the parameter α (if n and j are fixed).

Proposition 22 states that if n is large enough, then the functions $x \mapsto d_{n,j} + \eta_\alpha(x)/n$ are contractive and the fixed-point method yields the solution of (9).

Moreover, surprisingly for us, Newton's method applied to the equation $h_{\alpha,n,j}(x) = 0$ converges for *all* $n \ge 3$.

Theorem 3 convergence of Newton's method

Let $n \ge 3$, j be even, $1 \le j \le n$ and $y_{\alpha,n,j}^{(0)} \in \mathrm{cl}(I_{n,j})$. Define the sequence $(y_{\alpha,n,j}^{(m)})_{m=0}^{\infty}$ by the recursive formula

$$y_{\alpha,n,j}^{(m)} := y_{\alpha,n,j}^{(m-1)} - \frac{h_{\alpha,n,j}\left(y_{\alpha,n,j}^{(m-1)}\right)}{h'_{\alpha,n,j}\left(y_{\alpha,n,j}^{(m-1)}\right)} \quad (m \ge 1). \tag{11}$$

Then $(y_{\alpha,n,j}^{(m)})_{m=0}^{\infty}$ converges to $\vartheta_{\alpha,n,j}$. There is a constant $\mathscr{K}_2(\alpha)$ defined by (44) such that if $n > \sqrt{\pi\mathscr{K}_2(\alpha)/2}$, then for every m

$$\left|y_{\alpha,n,j}^{(m)} - \vartheta_{\alpha,n,j}\right| \le \frac{\pi}{n}\left(\frac{\pi\mathscr{K}_2(\alpha)}{2n^2}\right)^{2^m-1}. \tag{12}$$

We define $\Lambda_{\alpha,n} : [0,\pi] \to \mathbb{R}$ by

$$\Lambda_{\alpha,n}(x) := g(x) + \frac{g'(x)\eta_\alpha(x)}{n} + \frac{g'(x)\eta_\alpha(x)\eta'_\alpha(x) + \frac{1}{2}g''(x)\eta_\alpha(x)^2}{n^2}. \tag{13}$$

For j even, $1 \le j \le n$, we define $\lambda_{\alpha,n,j}^{\text{asympt}}$ by

$$\lambda_{\alpha,n,j}^{\text{asympt}} := \Lambda_{\alpha,n}\left(d_{n,j}\right). \tag{14}$$

Theorem 4 asymptotic expansion of the eigenvalues

There exists $C_1(\alpha) > 0$ such that for n large enough and j even, $1 \le j \le n$,

$$\left|\lambda_{\alpha,n,j} - \lambda_{\alpha,n,j}^{\text{asympt}}\right| \le \frac{C_1(\alpha)}{n^3}. \tag{15}$$

The asymptotic expansion (15) can be written as $\lambda_{\alpha,n,j} = \Lambda_{\alpha,n}(d_{n,j}) + O_\alpha(1/n^3)$, where the constant $C_1(\alpha)$ in the upper bound of $O_\alpha(1/n^3)$ depends on α, but does not depend on j or n.

Proposition 31 gives an alternative asymptotic expansion for $\lambda_{\alpha,n,j}$, with the points $j\pi/(n+1)$ instead of $d_{n,j}$.

Proposition 32 contains an asymptotic expansion of $\lambda_{\alpha,n,j}$ for small values of j, as j/n tends to 0. Notice that $\lambda_{\alpha,n,2}$ is the first non-zero eigenvalue of $L_{\alpha,n}$ and is known as the "spectral gap" of this matrix.

In the upcoming theorem we show an explicit formula (17) for the eigenvectors of $L_{\alpha,n}$ and asymptotic formulas for their norms; in these results we extend the domain of α to the strip $0 < \mathrm{Re}(\alpha) < 1$ of the complex plane, see (4). In the complex case we define $\varkappa_\alpha$ as $\mathrm{Re}(\alpha)/(1-\mathrm{Re}(\alpha))$. Formula (17) is a particular case of [32, Theorem 3.1].

For every x in $[0,\pi]$, we define

$$\nu_\alpha(x) := \frac{1-\mathrm{Re}(\alpha)}{2} g(x) - \frac{\mathrm{Re}(\alpha)}{2} g(\eta_\alpha(x)) + \frac{\mathrm{Re}(\alpha)-|\alpha|^2}{2} g(x-\eta_\alpha(x)) + 2|\alpha|^2. \tag{16}$$

Theorem 5 eigenvectors and their norms

Let $\alpha \in \mathbb{C}$, $0 < \mathrm{Re}(\alpha) < 1$. Then the vector $[1,\dots,1]^\top$ is an eigenvector of the matrix $L_{\alpha,n}$ associated to the eigenvalue $\lambda_{\alpha,n,1} = 0$. For every j, $2 \le j \le n$, and every k, $1 \le k \le n$, we define

$$v_{\alpha,n,j,k} := \sin(k\vartheta_{\alpha,n,j}) - (1-\overline{\alpha})\sin((k-1)\vartheta_{\alpha,n,j}) + \overline{\alpha}\sin((n-k)\vartheta_{\alpha,n,j}). \tag{17}$$

Then the vector $v_{\alpha,n,j} = [v_{\alpha,n,j,k}]_{k=1}^n$ with components (17) is an eigenvector of $L_{\alpha,n}$ associated to $\lambda_{\alpha,n,j}$. Moreover, if j is odd, then

$$\|v_{\alpha,n,j}\|_2 = |1-\alpha|\sqrt{\frac{n}{2}\lambda_{\alpha,n,j}}. \tag{18}$$

If j is even, then

$$\|v_{\alpha,n,j}\|_2 = \sqrt{n\nu_\alpha\left(\vartheta_{\alpha,n,j}\right)} + O_\alpha\left(\frac{1}{\sqrt{n}}\right), \tag{19}$$

with $O_\alpha\left(\frac{1}{\sqrt{n}}\right)$ uniformly on j.

3. Tridiagonal Toeplitz matrices with corner perturbations

Let $\delta, \varepsilon, \sigma, \tau$ be arbitrary complex parameters and $n \ge 3$. In this section, we consider the $n \times n$ matrix A_n obtained from the tridiagonal Toeplitz matrix with diagonals −1, 2, −1, substituting the components $(1,1)$, $(1,n)$, $(n,1)$, and (n,n) by $2-\delta$, $-\varepsilon$, $-\sigma$, and $2-\tau$, respectively. For example,

$$A_6 := \begin{bmatrix} 2-\delta & -1 & 0 & 0 & 0 & -\varepsilon \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -\sigma & 0 & 0 & 0 & -1 & 2-\tau \end{bmatrix}. \tag{20}$$

The study of more general tridiagonal symmetric Toeplitz matrices (with diagonals a_1, a_0, a_1 instead of –1,2,–1) with corner perturbations can be easily reduced to this case.

We are going to give formulas for the characteristic polynomial and eigenvectors of A_n. The results are not essentially new (see [10], [11], [32]), but we present them in a different form (with Chebyshev polynomials) and with other proofs.

We put $D_n(\lambda) := \det(\lambda I_n - A_n)$ and denote by T_n and U_n the Chebyshev polynomials of degree n of the first and second kind, respectively. The next proposition is a particular case of [11, Corollary 2.4]; it is also easy to prove directly expanding by cofactors.

Proposition 6

the characteristic polynomial of A_n

$$D_n(\lambda) = U_n\left(\frac{\lambda-2}{2}\right) + (\delta+\tau)U_{n-1}\left(\frac{\lambda-2}{2}\right) + (\delta\tau - \varepsilon\sigma)U_{n-2}\left(\frac{\lambda-2}{2}\right) + (-1)^{n+1}(\varepsilon+\sigma). \tag{21}$$

Corollary 7

If $\varepsilon = \delta$ and $\sigma = \tau = -\delta$, then $D_n(\lambda) = U_n((\lambda-2)/2)$. Therefore, the eigenvalues of A_n are $g(j\pi/(n+1))$ with j in $\{1,\dots,n\}$. The same situation holds for $\sigma = \delta$ and $\varepsilon = \tau = -\delta$.

If λ is an eigenvalue of A_n, we will search for an associated eigenvector $v = [v_k]_{k=1}^n$ as a linear combination of two geometric progressions:

$$v_k = G_1 z^k + G_2 z^{-k} \qquad (1 \le k \le n), \tag{22}$$

where z is a solution of the quadratic equation $z^2 + (\lambda-2)z + 1 = 0$. Equivalently, λ and z are related by

$$-z^{-1} + (2-\lambda) - z = 0. \tag{23}$$

Let $w := (\lambda I_n - A_n)v$. Formulas (22) and (23) easily imply that $w_k = 0$ for $2 \le k \le n-1$, and our goal is to find coefficients G_1 and G_2 such that $w_1 = 0$ and $w_n = 0$.

To take advantage of the symmetry between z and z^{-1}, we rewrite (22) in terms of Chebyshev polynomials:

$$v_k = \left(\frac{G_1+G_2}{2}\right)(z^k + z^{-k}) + \left(\frac{G_1-G_2}{2}\right)(z^k - z^{-k})$$
$$= (G_1+G_2)T_k\left(\frac{z+z^{-1}}{2}\right) + \frac{(G_1-G_2)(z-z^{-1})}{2}U_{k-1}\left(\frac{z+z^{-1}}{2}\right).$$

The system $w_1 = 0$ and $w_n = 0$ is equivalent to

$$\begin{aligned} \mathrm{a}_{\delta,\varepsilon,n}x + \mathrm{b}_{\delta,\varepsilon,n}y &= 0 \\ \mathrm{c}_{\sigma,\tau,n}x + \mathrm{d}_{\sigma,\tau,n}y &= 0, \end{aligned} \tag{24}$$

where $x := (G_1+G_2)/2$, $y := (G_1-G_2)/2$, and

$$\begin{aligned} \mathrm{a}_{\delta,\varepsilon,n} &:= 2\left(-1+\delta T_1\left(\tfrac{z+z^{-1}}{2}\right)+\varepsilon T_n\left(\tfrac{z+z^{-1}}{2}\right)\right), \\ \mathrm{b}_{\delta,\varepsilon,n} &:= (z-z^{-1})\left(\delta+\varepsilon U_{n-1}\left(\tfrac{z+z^{-1}}{2}\right)\right), \\ \mathrm{c}_{\sigma,\tau,n} &:= 2\left(\sigma T_1\left(\tfrac{z+z^{-1}}{2}\right)+\tau T_n\left(\tfrac{z+z^{-1}}{2}\right)-T_{n+1}\left(\tfrac{z+z^{-1}}{2}\right)\right), \\ \mathrm{d}_{\sigma,\tau,n} &:= (z-z^{-1})\left(\sigma+\tau U_{n-1}\left(\tfrac{z+z^{-1}}{2}\right)-U_n\left(\tfrac{z+z^{-1}}{2}\right)\right). \end{aligned} \tag{25}$$

In the next proposition we use the convention that $U_{-1}(t) := 0$.

Proposition 8

eigenvectors of A_n

Let $\lambda \in \mathbb{C} \setminus \{0,4\}$ be an eigenvalue of A_n. If $\mathrm{a}_{\delta,\varepsilon,n} \neq 0$ or $\mathrm{b}_{\delta,\varepsilon,n} \neq 0$, then the vector $v = [v_k]_{k=1}^n$ with components

$$v_k := (-1)^{k-1}\left(U_{k-1}\left(\tfrac{\lambda-2}{2}\right)+\delta U_{k-2}\left(\tfrac{\lambda-2}{2}\right)+(-1)^n\varepsilon U_{n-k-1}\left(\tfrac{\lambda-2}{2}\right)\right) \tag{26}$$

is an eigenvector of A_n associated to λ. If $\mathrm{c}_{\sigma,\tau,n} \neq 0$ or $\mathrm{d}_{\sigma,\tau,n} \neq 0$, then the vector $v = [v_k]_{k=1}^n$ with components

$$v_k := (-1)^{k-1}\left(\sigma U_{k-2}\left(\tfrac{\lambda-2}{2}\right)+(-1)^n\tau U_{n-k-1}\left(\tfrac{\lambda-2}{2}\right)+(-1)^n U_{n-k}\left(\tfrac{\lambda-2}{2}\right)\right) \tag{27}$$

is an eigenvector of A_n associated to λ.

Proof

The assumptions $\lambda \notin \{0,4\}$ and $z+z^{-1} = 2-\lambda$ imply that $z \notin \{-1,1\}$ and

$$T_n\left(\tfrac{\lambda-2}{2}\right) = (-1)^n\frac{z^n+z^{-n}}{2}, \qquad U_n\left(\tfrac{\lambda-2}{2}\right) = (-1)^n\frac{z^{n+1}-z^{-(n+1)}}{z-z^{-1}}. \tag{28}$$

A direct computation shows that

$$\mathrm{a}_{\delta,\varepsilon,n}\mathrm{d}_{\sigma,\tau,n} - \mathrm{b}_{\delta,\varepsilon,n}\mathrm{c}_{\sigma,\tau,n} = 2(-1)^n(z-z^{-1})D_n(\lambda).$$

Since λ is an eigenvalue of A_n, we get $D_n(\lambda) = 0$, and the linear homogeneous system (24) has non-trivial solutions (x,y). Namely, if $\mathrm{a}_{\delta,\varepsilon} \neq 0$ or $\mathrm{b}_{\delta,\varepsilon} \neq 0$, we put

$$x = \frac{\mathrm{b}_{\delta,\varepsilon}}{2(z-z^{-1})}, \qquad y = -\frac{\mathrm{a}_{\delta,\varepsilon}}{2(z-z^{-1})}.$$

Using (28) we simplify G_1 and G_2 to

$$\begin{aligned} G_1 &= x+y = \frac{\mathrm{b}_{\delta,\varepsilon,n}-\mathrm{a}_{\delta,\varepsilon,n}}{2(z-z^{-1})} = \frac{1-\delta z^{-1}-\varepsilon z^{-n}}{z-z^{-1}}, \\ G_2 &= x-y = \frac{\mathrm{b}_{\delta,\varepsilon,n}+\mathrm{a}_{\delta,\varepsilon,n}}{2} = \frac{-1+\delta z+\varepsilon z^n}{z-z^{-1}}. \end{aligned}$$

Hence, for every k, formula (22) converts in

$$v_k = \frac{z^k-z^{-k}}{z-z^{-1}} - \delta\,\frac{z^{k-1}-z^{-(k-1)}}{z-z^{-1}} + \varepsilon\,\frac{z^{n-k}-z^{-(n-k)}}{z-z^{-1}}, \tag{29}$$

which by (28) simplifies to (26). The linear independence of the geometric progressions $[z^k]_{k=1}^n$ and $[z^{-k}]_{k=1}^n$ assures that v is a non-zero vector. The proof of (27) is similar. □

Proposition 8 does not cover the situation when

$$\mathrm{a}_{\delta,\varepsilon,n} = \mathrm{b}_{\delta,\varepsilon,n} = \mathrm{c}_{\sigma,\tau,n} = \mathrm{d}_{\sigma,\tau,n} = 0. \tag{30}$$

We analyze this situation in the following remarks.

Remark 9

If $\lambda = 0$, i.e., $z = 1$, then (30) is equivalent to $\delta + \varepsilon = 1$ and $\sigma + \tau = 1$. The last two equalities imply that A_n is a laplacian complex matrix and $v = [1]_{k=1}^{n}$ is an eigenvector associated to λ.

Remark 10

If $\lambda = 4$, i.e., $z = -1$, then (30) is equivalent to $\delta + (-1)^n \varepsilon = 1$ and $(-1)^n \sigma - \tau = 1$. If these conditions are fulfilled, $v = [(-1)^k]_{k=1}^{n}$ is an eigenvector associated to λ.

Remark 11

If $\lambda \notin \{0, 4\}$, then (30) is equivalent to

$$\delta = \tau = \frac{U_{n-1}\left(\frac{z+z^{-1}}{2}\right)}{U_{n-2}\left(\frac{z+z^{-1}}{2}\right)}, \qquad \varepsilon = \sigma = -\frac{1}{U_{n-2}\left(\frac{z+z^{-1}}{2}\right)}.$$

In this case, every vector with components of the form (22) belongs to $\ker(\lambda I_n - A_n)$, and λ is an eigenvalue of multiplicity at least 2.

Remark 12

We have tested most formulas of this section in Sagemath using symbolic computations with polynomials over the variables $\delta, \varepsilon, \sigma, \tau, \lambda$, for every n with $3 \le n \le 20$. In particular, we have verified that if v is given by (26) and $w = (\lambda I_n - A_n)v$, then $w_n = (-1)^{n+1} D_n(\lambda)$. Analogously, if v is given by (27), then $w_1 = (-1)^n D_n(\lambda)$.

4. Eigenvalues' localization

In the incoming proposition, unlike the main part of the paper, we suppose that α is a complex parameter. We define $D_{\alpha,n}(\lambda)$ as the characteristic polynomial $\det(\lambda I_n - L_{\alpha,n})$, where $L_{\alpha,n}$ is the $n \times n$ complex laplacian matrix of the form (4).

Proposition 13 characteristic polynomial of complex laplacian matrices

For $n \ge 3$,

$$D_{\alpha,n}(\lambda) = (\lambda - 2\,\mathrm{Re}(\alpha))U_{n-1}\left(\frac{\lambda-2}{2}\right) - 2\,\mathrm{Re}(\alpha)U_{n-2}\left(\frac{\lambda-2}{2}\right) + 2(-1)^{n+1}\,\mathrm{Re}(\alpha). \tag{31}$$

Proof

This is a corollary of Proposition 6. □

Formula (31) implies a little miracle: $D_{\alpha,n} = D_{\mathrm{Re}(\alpha),n}$ for every complex α. Therefore, the eigenvalues of $L_{\alpha,n}$ are the same as the ones of the matrix $L_{\mathrm{Re}(\alpha),n}$. Since the latter matrix is hermitian, the eigenvalues are real. Hence, from now on we will suppose α to be a real number.

It turns out that $D_{\alpha,n}(\lambda)$ factorizes into a product of two polynomials of nearly the same degree. To join the cases when n is even and n is odd, we use the change of variables $\lambda = 4 - t^2$.

Proposition 14

For $n \ge 3$,

$$D_{\alpha,n}(4-t^2)=2(-1)^n\frac{p_n(t)q_{\alpha,n}(t)}{t},\tag{32}$$

where

$$p_n(t)=(t^2-4)U_{n-1}\left(\tfrac{t}{2}\right),\qquad q_{\alpha,n}(t)=(1-\alpha)T_n\left(\tfrac{t}{2}\right)+\alpha\tfrac{t}{2}U_{n-1}\left(\tfrac{t}{2}\right).$$

Proof

We will give a proof only for the case $n=2m$. The case $n=2m+1$ is similar. First, put $\lambda=2\omega+2$, hence $t^2=2-2\omega$. We apply the following elementary relations for Chebyshev polynomials:

$$\begin{aligned}&U_{2m-2}(\omega)=-U_{2m}(\omega)+2\omega U_{2m-1}(\omega),\\&U_{2m-1}(\omega)=2U_{m-1}(\omega)T_m(\omega),\\&U_{2m}(\omega)=2\omega U_{m-1}(\omega)T_m(\omega)+2T_m^2(\omega)-1,\\&T_m^2(\omega)-1=(\omega^2-1)U_{m-1}^2(\omega),\\&T_{2m}\left(\tfrac{t}{2}\right)=T_m\left(\tfrac{t^2-2}{2}\right),\quad U_{2m+1}\left(\tfrac{t}{2}\right)=tU_m\left(\tfrac{t^2-2}{2}\right).\end{aligned}$$

Thereby we obtain the next chain of equalities:

$$\begin{aligned}&D_{\alpha,2m}(2\omega+2)=2\Big(\alpha U_{2m}(\omega)+(\omega+1-\alpha-2\alpha\omega)U_{2m-1}(\omega)-\alpha\Big)\\&=4\Big((\omega+1)(1-\alpha)U_{m-1}(\omega)T_m(\omega)+\alpha(T_m^2(\omega)-1)\Big)\\&=4(\omega+1)U_{m-1}(\omega)\Big((1-\alpha)T_m(\omega)-\alpha(1-\omega)U_{m-1}(\omega)\Big),\end{aligned}$$

and we arrive at (32). □

The factorization (32) after the change of variable $t=2\cos(x/2)$ reads as

$$D_{\alpha,n}(g(x))=D_{\alpha,n}(4-(2\cos(x/2))^2)=(-1)^n\frac{p_n(2\cos(x/2))q_{\alpha,n}(2\cos(x/2))}{\cos(x/2)},\tag{33}$$

where

$$p_n(2\cos(x/2))=-4\sin\tfrac{x}{2}\sin\tfrac{nx}{2},\qquad q_{\alpha,n}(2\cos(x/2))=(1-\alpha)\cos\tfrac{nx}{2}+\alpha\cos\tfrac{x}{2}\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}},$$

or

$$D_{\alpha,n}(g(x))=(-1)^{n+1}\frac{4\sin\frac{x}{2}\sin\frac{nx}{2}}{\cos\frac{x}{2}}\left((1-\alpha)\cos\tfrac{nx}{2}+\alpha\cos\tfrac{x}{2}\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right).\tag{34}$$

The polynomial p_n does not depend on α, and its zeros are easy to find.

Proposition 15

trivial eigenvalues of $L_{\alpha,n}$

For every $n\geq 3$ and every even k with $0\leq k\leq n-1$, the number $g(k\pi/n)$ is an eigenvalue of $L_{\alpha,n}$.

Proof

The number $t=2\cos(k\pi/(2n))$, with k as in the hypothesis, is a zero of p_n. It corresponds to the eigenvalue $\lambda=4-t^2=g(k\pi/n)$, since $g(x)=4-(2\cos(x/2))^2$. □

We already have an explicit formula for $\lfloor (n+1)/2 \rfloor$ eigenvalues of $L_{\alpha,n}$. The remaining ones correspond to the zeros of the polynomial $q_{\alpha,n}$. To analyze their localization, we first compute the values of $q_{\alpha,n}$ at the points $2\cos(j\pi/(2n))$ which correspond to the uniform mesh $j\pi/n$, $j = 0, \ldots, n$.

The next lemma is easily proven by direct computations.

Lemma 16

For every j with $1 \le j \le n-1$,

$$q_{\alpha,n}\left(2\cos\frac{j\pi}{2n}\right) = \begin{cases} (1-\alpha)(-1)^{\frac{j}{2}}, & if\ j\ is\ even, \\ \alpha\cot\frac{j\pi}{2n}(-1)^{\frac{j-1}{2}}, & if\ j\ is\ odd. \end{cases}$$

Moreover,

$$q_{\alpha,n}(0) = \begin{cases} 0, & if\ n\ is\ odd, \\ (-1)^{\frac{n}{2}}(1-\alpha), & if\ n\ is\ even, \end{cases} \qquad q_{\alpha,n}(2) = (1-\alpha) + \alpha n.$$

We observe that if n is even, then $p_n(0) = 0$, and if n is odd, then $q_{\alpha,n}(0) = 0$. However, $t = 0$ may not be a zero of $D_{\alpha,n}(4-t^2)$ because of the factor $1/t$ in (32). This leads us to the next elementary lemma.

Lemma 17

If n is odd, then

$$\lim_{t\to 0^+} \frac{2q_{\alpha,n}(t)}{t} = (-1)^{\frac{n-1}{2}}\Big(\alpha + (1-\alpha)n\Big),$$

and if n is even, then

$$\lim_{t\to 0^+} \frac{2p_n(t)}{t} = 4(-1)^{\frac{n}{2}}n.$$

Proof of Theorem 1

Let $1 \le j \le n$. If j is odd, then (5) follows by Proposition 15.

We consider the quotient $q_{\alpha,n}(2\cos(x/2))/(2\cos(x/2))$ from factorization (33). Lemma 16, Lemma 17 imply that this expression changes its sign in the intervals $I_{n,j}$, where j is even. By the intermediate value theorem, we have (6). □

Theorem 1 implies immediately that for every $0 < \alpha < 1$ and for every y in $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{\#\{j \in \{1,\ldots,n\}:\ \lambda_{\alpha,n,j} \le y\}}{n} = \frac{\mu(\{x \in [0,\pi]: g(x) \le y\})}{\pi},$$

i.e., the eigenvalues of $L_{\alpha,n}$ are asymptotically distributed as the function g on $[0,\pi]$.

5. Main equation

In this section we reduce the computation of the non-trivial eigenvalues to the solution of the "main equation" (9). We recall it here:

$$x = d_{n,j} + \frac{\eta_\alpha(x)}{n}.$$

Proof of Theorem 2

Recall that j is even. In the proof of Theorem 1 we have seen that $\vartheta_{\alpha,n,j}$ belongs to $I_{n,j}$ and is the unique solution of the equation $q_{\alpha,n}(2\cos(x/2)) = 0$. This is equivalent to the following one (see also (33)):

$$\tan \frac{nx}{2} = -\frac{1-\alpha}{\alpha} \tan \frac{x}{2}. \tag{35}$$

Applying arctan to both sides of (35) we transform it to

$$nx = j\pi - 2\arctan\left(\frac{1-\alpha}{\alpha} \tan \frac{x}{2}\right).$$

Finally, since $\pi/2 - \arctan(u) = \arctan(1/u)$, we obtain (9). □

Fig. 4 shows the plots of both sides of (35) for some α in $(0,1)$. We see that the intersections really take place in the intervals given in Theorem 1.

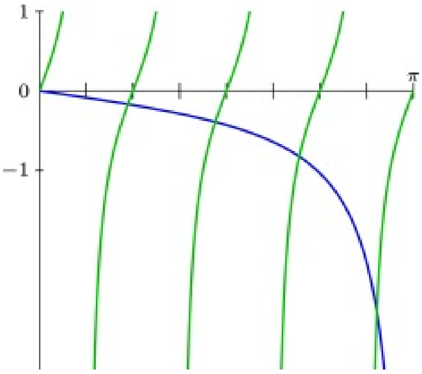

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Fig. 4. The left-hand side (green) and right-hand side (blue) of (35) for α=0.7 and n=8; the scales of the axes are different.

Recall that $h_{\alpha,n,j}$ is defined by (10). Obviously, (9) is equivalent to $h_{\alpha,n,j}(x) = 0$.

Proposition 18

Let $n \geq 3$ and j be even with $1 \leq j \leq n$. Then $h_{\alpha,n,j}$ changes its sign exactly once in $I_{n,j}$.

Proof

Indeed,

$$h_{\alpha,n,j}((j-1)\pi/n) = -\eta_\alpha((j-1)\pi/n) < 0,$$

$$h_{\alpha,n,j}(j\pi/n) = \pi - \eta_\alpha(j\pi/n) > 0,$$

and $h_{\alpha,n,j}$ is strictly increasing. □

Remark 19

If $\alpha = 1/2$, then $\varkappa_{\frac{1}{2}} = 1$ and $\eta_\alpha(x) = \pi - x$. In this case equation (9) yields explicit formulas for the eigenvalues $\lambda_{\alpha,n,j}$ with even values of j:

$$\vartheta_{\alpha,n,j} = \frac{j\pi}{n+1}, \qquad \lambda_{\alpha,n,j} = g\left(\frac{j\pi}{n+1}\right).$$

In the following proposition, unlike in the other parts of this paper, we fix n and j and treat α as a variable running through the closed interval $[0,1]$. Formally, we define $\Psi_{n,j}:[0,1]\to[0,4]$ by

$$\Psi_{n,j}(\alpha):=\lambda_{\alpha,n,j}.$$

Proposition 20

dependence of the eigenvalues on the parameter α

Let $n\ge 3$ and j be even, with $1\le j\le n$. Then $\Psi_{n,j}$ is continuous and strictly increasing on $[0,1]$. In particular,

$$\lim_{\alpha\to 0^+}\lambda_{\alpha,n,j}=\lambda_{0,n,j}=g\left(\frac{(j-1)\pi}{n}\right), \tag{36}$$

$$\lim_{\alpha\to 1^-}\lambda_{\alpha,n,j}=\lambda_{1,n,j}=g\left(\frac{j\pi}{n}\right). \tag{37}$$

Proof

It is well known that the functions $A\mapsto\lambda_j(A)$ are Lipschitz continuous on the space of the hermitian matrices provided with the operator norm, see [18, Weyl's Theorem 4.3.1 and Problem 4.3.P1]. As a consequence, $\Psi_{n,j}$ is continuous on $[0,1]$.

To analyze the monotonicity, we will apply to the main equation some ideas from the implicity function theorem. Define $\Theta_{n,j}:(0,\pi)\to\mathbb{R}$ and $H_{n,j}:(0,1)\times(0,\pi)\to\mathbb{R}$ by

$$\Theta_{n,j}(\alpha):=\vartheta_{\alpha,n,j},\qquad H_{n,j}(\alpha,x):=h_{\alpha,n,j}(x)=nx-(j-1)\pi-\eta_\alpha(x).$$

Compute the partial derivatives of $H_{n,j}$ with respect to the first and second argument:

$$(D_1H_{n,j})(\alpha,x)=-\frac{2\tan\frac{x}{2}}{\alpha^2+(1-\alpha)^2\tan^2\frac{x}{2}}<0,\qquad (D_2H_{n,j})(\alpha,x)=n-\eta'_\alpha(x)>n.$$

Since $H_{n,j}(\alpha,\Theta_{n,j}(\alpha))=0$, we conclude that $\Theta_{n,j}$ is differentiable on $(0,1)$, and

$$\Theta'_{n,j}(\alpha)=-\frac{(D_1H_{n,j})(\alpha,\Theta_{n,j}(\alpha))}{(D_2H_{n,j})(\alpha,\Theta_{n,j}(\alpha))}>0.$$

Hence, the functions $\Theta_{n,j}$ and $\Psi_{n,j}=g\circ\Theta_{n,j}$ are strictly increasing on $(0,1)$. Now the continuity of $\Psi_{n,j}$ implies that this function is strictly increasing on $[0,1]$. □

Fig. 5 shows the eigenvalues $\lambda_{\alpha,n,j}=g(\vartheta_{\alpha,n,j})$ for $\alpha=1/3$ and $\alpha=4/5$, with $n=10$. One can observe the localization of $\vartheta_{\alpha,n,j}$ in $I_{n,j}$ for even values of j and the monotone dependence on α.

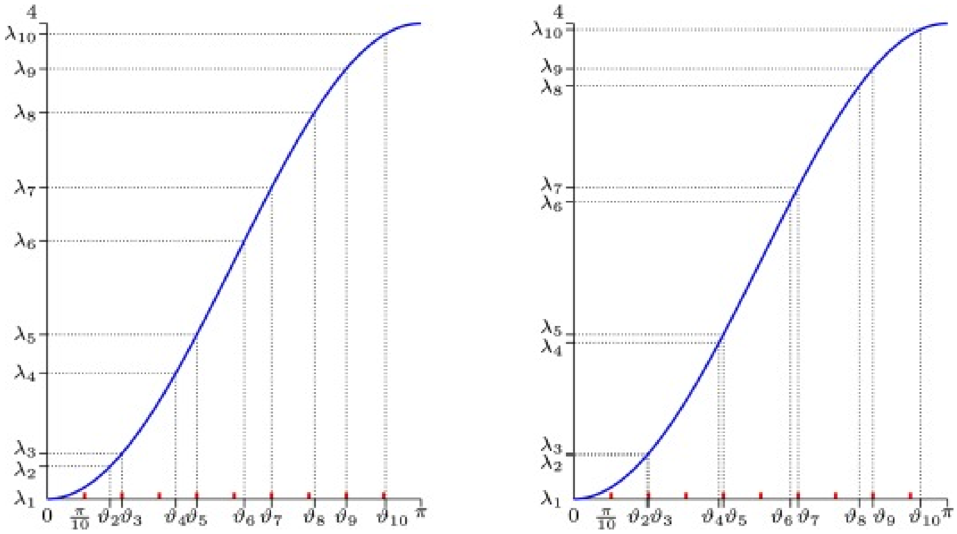

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Fig. 5. The values $\vartheta_{\alpha,n,j}$ and $\lambda_{\alpha,n,j}$ for $\alpha=1/3$, $n=10$ (left) and $\alpha=4/5$, $n=6$ (right); the red marks on the horizontal axis correspond to $k\pi/10$, $1\le k\le 9$.

6. Solving the main equation by the fixed-point method

We recall that $\eta_{\alpha,n}$ and $\varkappa_\alpha$ are defined by (7) and (8), respectively, and that η_α does not depend of n. Here are other equivalent formulas for η_α:

$$\eta_\alpha(x) = \pi - 2\arctan\left(\frac{1-\alpha}{\alpha}\tan\frac{x}{2}\right), \tag{38}$$

$$\eta_\alpha(x) = 2\arcsin\frac{\varkappa_\alpha\cos\frac{x}{2}}{\sqrt{\sin^2\frac{x}{2}+\varkappa_\alpha^2\cos^2\frac{x}{2}}}, \tag{39}$$

$$\eta_\alpha(x) = 2\arcsin\frac{\sqrt{2}\,\alpha\cos\frac{x}{2}}{\sqrt{(2\alpha^2-2\alpha+1)+(2\alpha-1)\cos(x)}}. \tag{40}$$

We notice that (7) is more convenient to use if x is close to π, while (38) is better for x close to 0. The first two derivatives of $\eta_{\alpha,n}$ are

$$\eta'_\alpha(x) = -\frac{\varkappa_\alpha(1+\tan^2\frac{x}{2})}{\varkappa_\alpha^2+\tan^2\frac{x}{2}}, \tag{41}$$

$$\eta'_\alpha(x) = -\frac{\varkappa_\alpha(1+\cot^2\frac{x}{2})}{1+\varkappa_\alpha^2\cot^2\frac{x}{2}}, \tag{42}$$

$$\eta''_\alpha(x) = \frac{(\varkappa_\alpha^2-1)\tan\frac{x}{2}}{\varkappa_\alpha^2+\tan^2\frac{x}{2}}\,\eta'_\alpha(x). \tag{43}$$

The incoming proposition gives some upper bounds for η'_α and η''_α for every α in $(0,1)$, involving the following numbers:

$$\mathscr{K}_1(\alpha) := \max\left\{\varkappa_\alpha, \frac{1}{\varkappa_\alpha}\right\}, \qquad \mathscr{K}_2(\alpha) := \frac{|\varkappa_\alpha^2-1|}{2\varkappa_\alpha}\mathscr{K}_1(\alpha) = \frac{\mathscr{K}_1^2(\alpha)-1}{2}. \tag{44}$$

Proposition 21

Each derivative of η_α is a bounded function on $(0,\pi)$. In particular,

$$\sup_{0<x<\pi} |\eta'_\alpha(x)| = \mathscr{K}_1(\alpha), \tag{45}$$

$$\sup_{0<x<\pi} |\eta''_\alpha(x)| \le \mathscr{K}_2(\alpha). \tag{46}$$

Proof

In order to prove (45), we rewrite (41) as follows:

$$\eta'_\alpha(x) = -\varkappa_\alpha \left(1 + \frac{1-\varkappa_\alpha^2}{\varkappa_\alpha^2+\tan^2\frac{x}{2}}\right) \qquad (x \in (0,\pi)). \tag{47}$$

We notice that $\tan^2(x/2)$ increases from 0 to ∞ as x goes from 0 to π. If $0 < \alpha \le 1/2$, then $\varkappa_\alpha \le 1$, and η'_α increases taking values from $\eta'_\alpha(0) = -\varkappa_\alpha^{-1}$ to $\eta'_\alpha(\pi) = -\varkappa_\alpha$. If $1/2 < \alpha < 1$, then η'_α decreases. In both cases, the maximal value of $|\eta'_\alpha|$ is reached at one of the points 0 or π. This proves (45).

For the second derivative of η''_α, from (43) we get

$$|\eta''_\alpha(x)| = \frac{\tan\frac{x}{2}}{\varkappa_\alpha^2+\tan^2\frac{x}{2}} |\varkappa_\alpha^2 - 1||\eta'_\alpha(x)| \le \frac{|\varkappa_\alpha^2-1|}{2|\varkappa_\alpha|}\mathscr{K}_1(\alpha) \qquad (x \in (0,\pi)).$$

This is exactly (46).

For the higher derivatives of $\eta_{\alpha,j}$, the explicit estimates are too tedious, and we propose the following argument. By (41), η'_α is analytic in a neighborhood of x, for any x in $(0,\pi)$. Even more, η'_α has an analytic extension in some neighborhoods of the points 0 and π. Hence, $\eta'_{\alpha,j}$ has an analytic extension to a certain open set in the complex plane containing the segment $[0,\pi]$. Therefore, each derivative of this function is bounded on $(0,\pi)$. □

For every j, $1 \le j \le n$, we define the function $f_{\alpha,n,j} : [0,\pi] \to \mathbb{R}$ by

$$f_{\alpha,n,j}(x) := d_{n,j} + \frac{\eta_\alpha(x)}{n}, \tag{48}$$

i.e., $f_{\alpha,n,j}(x) = ((j-1)\pi + \eta_\alpha(x))/n$. Hence (9) can be written as $\vartheta_{\alpha,n,j} = f_{\alpha,n,j}(\vartheta_{\alpha,n,j})$.

Proposition 22

Let $n > \mathscr{K}_1(\alpha)$, and let j be even, $1 \le j \le n$. Then $f_{\alpha,n,j}$ is contractive in $\operatorname{clos}(I_{n,j})$. Its fixed point belongs to $I_{n,j}$ and coincides with $\vartheta_{\alpha,n,j}$.

Proof

Since η_α takes values in $[0,\pi]$, for every x in $\operatorname{clos}(I_{n,j})$ we get

$$\frac{(j-1)\pi}{n} \le \frac{(j-1)\pi + \eta_\alpha(x)}{n} \le \frac{j\pi}{n},$$

i.e., $f_{\alpha,n,j}(x) \in \operatorname{clos}(I_{n,j})$. By Proposition 21, η'_α is bounded by $\mathscr{K}_1(\alpha)$, hence

$$\left|f'_{\alpha,n,j}(x)\right| \le \frac{\mathscr{K}_1(\alpha)}{n} < 1.$$

This implies that $f_{\alpha,n,j}$ is a contractive function on $\operatorname{clos}(I_{n,j})$. Then, by the Banach fixed point theorem, $f_{\alpha,n,j}$ has a unique fixed point, and by Theorem 2 it coincides with $\vartheta_{\alpha,n,j}$ and belongs to $I_{n,j}$. □

Corollary 23

Let $n > \mathscr{K}_1(\alpha)$, j be even, $1 \le j \le n$, and $x^{(0)}_{\alpha,n,j}$ be an arbitrary point in $\operatorname{clos}(I_{n,j})$. Define the sequence $\left(x^{(m)}_{\alpha,n,j}\right)_{m=0}^{\infty}$ by

$$x_{\alpha,n,j}^{(m)} := f_{\alpha,n,j}\left(x_{\alpha,n,j}^{(m-1)}\right) \qquad (m \geq 1).$$

Then

$$\left|x_{\alpha,n,j}^{(m)} - \vartheta_{\alpha,n,j}\right| \leq \frac{\pi}{n}\left(\frac{\mathscr{K}_1(\alpha)}{n}\right)^m \qquad (m \geq 0).$$

Proof

Follows from Proposition 22 and Banach fixed point theorem. □

7. Newton's method for convex functions

In this section we recall some sufficient conditions for the convergence of Newton's method. Assume that $a, b \in \mathbb{R}$ with $a < b$; f is differentiable and $f' > 0$ on $[a, b]$; there exists c in $[a, b]$ such that $f(c) = 0$; $y^{(0)}$ is a point in $[a, b]$ and the sequence $(y^{(m)})_{m=0}^{\infty}$ is defined (when possible) by the recurrence relation

$$y^{(m+1)} = y^{(m)} - \frac{f(y^{(m)})}{f'(y^{(m)})}. \tag{49}$$

Obviously, if $y^{(m)} = c$ for some m, then the sequence is constant starting from this moment.

In general, the sufficient conditions for Newton's method are quite complicated (see, for example, Kantorovich's theorem). Nevertheless, it is well known that Newton's method converges for convex functions, when the initial point is chosen from the “correct” side of the root ([26, Section 22, Problem 14] and [2, Theorem 2.2]). In the following proposition we show an upper bound for the linear convergence in this case.

Proposition 24 linear convergence of Newton's method for convex functions

If f is convex on $[a, b]$, $c \leq y^{(0)} \leq b$, then $y^{(m)}$ belongs to $[c, b]$ for every $m \geq 0$, the sequence $(y^{(m)})_{m=0}^{\infty}$ decreases and converges to c, with

$$y^{(m)} - c \leq (b - a)\left(1 - \frac{f'(a)}{f'(b)}\right)^m. \tag{50}$$

Proof

Reasoning by induction, suppose that $m \geq 1$ and $b \geq y^{(m)} \geq c$. By the mean value theorem, there exists $\xi_m \in [c, y^{(m)}]$ such that $f(y^{(m)}) - f(c) = f'(\xi_m)(y^{(m)} - c)$, hence

$$f(y^{(m)}) = (y^{(m)} - c) f'(\xi_m). \tag{51}$$

Combining (49) with (51) we obtain that

$$y^{(m+1)} - c = y^{(m)} - \frac{f(y^{(m)})}{f'(y^{(m)})} - c = \left(y^{(m)} - c\right)\left(1 - \frac{f'(\xi_m)}{f'(y^{(m)})}\right). \tag{52}$$

Since f' is positive and increasing on $[a, b]$,

$$0 \leq y^{(m+1)} - c \leq \left(y^{(m)} - c\right)\left(1 - \frac{f'(a)}{f'(b)}\right). \tag{53}$$

This yields (50) and the convergence of the sequence. □

The next proposition provides a sufficient convergence condition, when starting from the “bad” side of the root. Then $y^{(1)}$ is on the “good” side of the root and Proposition 24 can be applied to the sequence $(y^{(m)})_{m=1}^{\infty}$.

Proposition 25

Suppose that f is convex on $[a,b]$, $a \le y^{(0)} < c$, *and*

$$a - \frac{f(a)}{f'(a)} \le b. \tag{54}$$

Then $y^{(1)}$ *belongs to* $[c,b]$.

Proof

Since f is convex, its graph is above the tangent lines at the points $(a, f(a))$ and $(y^{(0)}, f(y^{(0)}))$. In particular,

$$f(y^{(0)}) \ge f(a) + f'(a)(y^{(0)} - a), \qquad 0 = f(c) \ge f(y^{(0)}) + f'(y^{(0)})(c - y^{(0)}).$$

Moreover, $f(y^{(0)}) < 0$ and $f'(y^{(0)}) \ge f'(a) > 0$. Hence,

$$c \le y^{(1)} = y^{(0)} - \frac{f(y^{(0)})}{f'(y^{(0)})} \le y^{(0)} - \frac{f(a)+f'(a)(y^{(0)}-a)}{f'(a)} = a - \frac{f(a)}{f'(a)} \le b.$$

□

The following fact is well known [2, Theorem 2.1].

Proposition 26

Let $f \in C^2([a,b])$. *Suppose that* $(b-a)M < 1$, *where*

$$M := \frac{\max_{t\in[a,b]} |f''(t)|}{2 \min_{t\in[a,b]} |f'(t)|}.$$

Assume that $y^{(m)}$ *is well defined and belongs to* $[a,b]$ *for every m. Then* $y^{(m)}$ *converges to c as m tends to* ∞, *and for every m*

$$|y^{(m)} - c| \le ((b-a)M)^{2^m-1}(b-a). \tag{55}$$

Idea of the proof

Let $m \ge 0$. By Taylor's formula, there exists $\nu \in [c, y^{(m)}]$ such that

$$0 = f(c) = f(y^{(m)}) + f'(y^{(m)})\left(c - y^{(m)}\right) + \tfrac{1}{2} f''(\nu)\left(c - y^{(m)}\right)^2.$$

It follows easily that $|y^{(m+1)} - c| \le M(c - y^{(m)})^2$. Now (55) is obtained by induction. □

Remark 27 Newton's method for concave functions

Analogs of Proposition 24, Proposition 25 hold if f is a concave function. In this case, each of the following two conditions is sufficient for the convergence:

- $a \le y^{(0)} \le c$,
- $c < y^{(0)} \le b$ and $b - f(b)/f'(b) \ge a$.

Instead of repeating the corresponding proofs with obvious modifications, one can pass to the function $x \mapsto -f(-x)$.

8. Solving the main equation by Newton's method

Recall that $h_{\alpha,n,j}$ is defined by (10). In this section we prove that the equation $h_{\alpha,n,j}(x)=0$, which is equivalent to the main equation, can be solved by Newton's method for every $n\geq 3$.

Remark 19 shows that the eigenvalues can be exactly computed if $\alpha=1/2$, hence this case could be omitted in the next propositions.

Proposition 28 linear convergence of Newton's method applied to the main equation

For every $n\geq 3$ and every j be even, $1\leq j\leq n$ and $y_{\alpha,n,j}^{(0)}\in I_{n,j}$, the sequence $(y_{\alpha,n,j}^{(m)})_{m=0}^{\infty}$, defined by (11), converges to $\vartheta_{\alpha,n,j}$. The convergence is at least linear:

$$\left|y_{\alpha,n,j}^{(m)}-\vartheta_{\alpha,n,j}\right|\leq\frac{\pi}{n}\gamma_{\alpha,n}^{m-1}, \tag{56}$$

where

$$\gamma_{\alpha,n}:=\frac{|2\alpha-1|}{\alpha(1-\alpha)n+|2\alpha-1|}. \tag{57}$$

Proof

We start with the case $1/2\leq\alpha\leq 1$. By the proof of Proposition 21, η_α is analytic in $\mathrm{cl}(I_{n,j})$), and η'_α decreases on $[0,\pi]$ taking values $\eta'_\alpha(0)=-\varkappa_\alpha^{-1}$ to $\eta'_\alpha(\pi)=-\varkappa_\alpha$. Therefore, $h_{\alpha,n,j}$ is analytic and convex on $[0,\pi]$, and

$$1-\frac{n-\eta'_\alpha\left(\frac{(j-1)\pi}{n}\right)}{n-\eta'_\alpha\left(\frac{j\pi}{n}\right)}\leq 1-\frac{n-\eta'_\alpha(0)}{n-\eta'_\alpha(\pi)}=\frac{2\alpha-1}{\alpha(1-\alpha)n+\alpha^2}\leq\frac{2\alpha-1}{\alpha(1-\alpha)n+2\alpha-1}.$$

If $y_{\alpha,n,j}^{(0)}\geq\vartheta_{\alpha,n,j}$, then Proposition 24 yields the convergence and (56).

For $y_{\alpha,n,j}^{(0)}<\vartheta_{\alpha,n,j}$, we have to verify the condition (54) from Proposition 25. In effect,

$$\frac{(j-1)\pi}{n}-\frac{h_{\alpha,n,j}\left(\frac{(j-1)\pi}{n}\right)}{h'_{\alpha,n,j}\left(\frac{(j-1)\pi}{n}\right)}=\frac{(j-1)\pi}{n}+\frac{\eta_{\alpha,n,j}\left(\frac{(j-1)\pi}{n}\right)}{n-\eta'_{\alpha,n,j}\left(\frac{(j-1)\pi}{n}\right)}\leq\frac{j\pi}{n}.$$

Since $y_{\alpha,n,j}^{(1)}\geq\vartheta_{\alpha,n,j}$, after applying $m-1$ steps of the algorithm we get (56).

For $0\leq\alpha\leq 1/2$, $h_{\alpha,n,j}$ is concave, and the proof of the linear convergence is similar (see Remark 27). In particular, if $y_{\alpha,n,j}^{(0)}>\vartheta_{\alpha,n,j}$, then

$$\frac{j\pi}{n}-\frac{h_{\alpha,n,j}\left(\frac{j\pi}{n}\right)}{h'_{\alpha,n,j}\left(\frac{j\pi}{n}\right)}=\frac{j\pi}{n}-\frac{\pi-\eta_\alpha\left(\frac{j\pi}{n}\right)}{n-\eta'_\alpha\left(\frac{j\pi}{n}\right)}\geq\frac{(j-1)\pi}{n}.$$

□

Proof of Theorem 3

The first part of Theorem follows from Proposition 28. Now we suppose that $0<\alpha<1$ and $n>\sqrt{\pi\mathscr{K}_2(\alpha)/2}$. Since $\eta'_\alpha<0$ and $|\eta''_\alpha|$ is bounded by $\mathscr{K}_2(\alpha)$,

$$M_{\alpha,n,j}:=\frac{1}{2}\sup_{0<x,y<\pi}\left|\frac{h''_{\alpha,n,j}(x)}{h'_{\alpha,n,j}(y)}\right|=\frac{1}{2n}\sup_{0<x,y<\pi}\left|\frac{\eta''_\alpha(x)}{1-\frac{\eta'_\alpha(y)}{n}}\right|\leq\frac{\mathscr{K}_2(\alpha)}{2n}.$$

Therefore, $\frac{\pi}{n}M_{\alpha,n,j}\leq\frac{\pi\mathscr{K}_2(\alpha)}{2n^2}<1$, the conditions in Proposition 26 are fulfilled, and we obtain (12). □

The upper bound (56) allows us to compute "a priori" the number of steps that will be sufficient to achieve a desired precision. Namely, if

$$m > \frac{p+\log_2 \frac{\pi}{2n}}{\log_2 \frac{1}{\gamma_{\alpha,n}}} + 1, \tag{58}$$

then $|y_{\alpha,n,j}^{(m)} - \vartheta_{\alpha,n,j}| < 2^{-p}$. In fact, after a few iterations, the linear convergence transforms into quadratic convergence, hence reducing the number of iterations.

9. Asymptotic formulas for the eigenvalues

Proposition 29

Let $n \geq 3$ *and* j *be even with* $1 \leq j \leq n$. *Then*

$$\left|\vartheta_{\alpha,n,j} - \left(d_{n,j} + \frac{\eta_\alpha(d_{n,j})}{n}\right)\right| \leq \frac{\pi \mathcal{K}_1(\alpha)}{n^2}. \tag{59}$$

Proof

Theorem 2 assures that $|\vartheta_{\alpha,n,j} - d_{n,j}| \leq \frac{\pi}{n}$. Hence, by the mean value theorem and formula (45),

$$|\eta_\alpha(\vartheta_{\alpha,n,j}) - \eta_\alpha(d_{n,j})| \leq \|\eta'_\alpha\|_\infty \ |\vartheta_{\alpha,n,j} - d_{n,j}| \leq \frac{\pi \mathcal{K}_1(\alpha)}{n}.$$

Using (9) we obtain (59). □

Proposition 30

There exists $C_1(\alpha) > 0$ *such that for every* $n \geq 3$ *and every* j *even with* $1 \leq j \leq n$,

$$\vartheta_{\alpha,n,j} = d_{n,j} + \frac{\eta_\alpha(d_{n,j})}{n} + \frac{\eta_\alpha(d_{n,j})\eta'_\alpha(d_{n,j})}{n^2} + r_{\alpha,n,j}, \tag{60}$$

where $|r_{\alpha,n,j}| \leq \frac{C_1(\alpha)}{n^3}$.

Proof

Proposition 29 implies that

$$\vartheta_{\alpha,n,j} = d_{n,j} + \frac{\eta_\alpha(d_{n,j})}{n} + O_\alpha\left(\frac{1}{n^2}\right).$$

Substitute this expression into the right-hand side of (9):

$$\vartheta_{\alpha,n,j} = d_{n,j} + \frac{\eta_\alpha\left(d_{n,j} + \frac{\eta_\alpha(d_{n,j})}{n} + O_\alpha\left(\frac{1}{n^2}\right)\right)}{n}.$$

Expanding η_α by Taylor's formula around $d_{n,j}$ with two exact term and estimating the residue term with Proposition 21 we obtain the desired result. □

Proof of Theorem 4

This theorem follows from Proposition 30: we just evaluate g at the expression (60) and expand it by Taylor's formula around $d_{n,j}$. □

In a similar manner, iterating in the main equation (9), we could obtain asymptotic expansions with more terms; see [3, (3.9)] for the asymptotic expansions up to n^{-5}.

There are other forms of the asymptotic expansions for $\lambda_{\alpha,n,j}$. Adding x to both sides of the equation $nx = (j-1)\pi + \eta_\alpha(x)$ and dividing it over $n+1$, we arrive at the following equivalent form of the main equation:

$$x = \frac{j\pi + \tilde{\eta}_\alpha(x)}{n+1}, \quad \text{where} \quad \tilde{\eta}_\alpha(x) := \eta_\alpha(x) + x - \pi = 2\arctan\frac{(\varkappa_\alpha - 1)\cot\frac{x}{2}}{1 + \varkappa_\alpha \cot^2\frac{x}{2}}. \tag{61}$$

After that, similarly to Proposition 30 and Theorem 4, we obtain the next result.

Proposition 31

There exist $C_2(\alpha) > 0$ and $C_3(\alpha) > 0$ such that for every $n \geq 3$ and every j even with $1 \leq j \leq n$,

$$\vartheta_{\alpha,n,j} = \frac{j\pi}{n+1} + \frac{\tilde{\eta}_\alpha\left(\frac{j\pi}{n+1}\right)}{n+1} + \frac{\tilde{\eta}_\alpha\left(\frac{j\pi}{n+1}\right)\tilde{\eta}'_\alpha\left(\frac{j\pi}{n+1}\right)}{(n+1)^2} + \tilde{r}_{\alpha,n,j}, \tag{62}$$

$$\begin{aligned}\lambda_{\alpha,n,j} &= g\left(\frac{j\pi}{n+1}\right) + \frac{g'\left(\frac{j\pi}{n+1}\right)\tilde{\eta}_\alpha\left(\frac{j\pi}{n+1}\right)}{n+1} \\ &\quad + \frac{g'\left(\frac{j\pi}{n+1}\right)\tilde{\eta}_\alpha\left(\frac{j\pi}{n+1}\right)\tilde{\eta}'_\alpha\left(\frac{j\pi}{n+1}\right) + \frac{1}{2}g''\left(\frac{j\pi}{n+1}\right)\tilde{\eta}_\alpha\left(\frac{j\pi}{n+1}\right)^2}{(n+1)^2} + \widetilde{R}_{\alpha,n,j},\end{aligned} \tag{63}$$

where $|\tilde{r}_{\alpha,n,j}| \leq \frac{C_2(\alpha)}{n^3}$ and $|\widetilde{R}_{\alpha,n,j}| \leq \frac{C_3(\alpha)}{n^3}$.

Numerical experiments show that (63) is more precise than (14), especially for α close to 1/2, but the errors are almost the same for α close to 1. Moreover, $\tilde{\eta}_\alpha$ is more complicated than η_α ($\tilde{\eta}_\alpha$ has two intervals of monotonicity), and the denominator n naturally appears in the formula (5) for $\lambda_{\alpha,n,j}$ with odd j.

In the incoming proposition we obtain a simplified asymptotic formula for the eigenvalues $\lambda_{\alpha,n,j}$ as j/n tends to zero.

Proposition 32

Let α be a fixed number in $(0,1)$. Then $\lambda_{\alpha,n,j}$ has the following asymptotic expansion as j/n tends to 0:

$$\lambda_{\alpha,n,j} = \frac{j^2\pi^2}{n^2} - \frac{2j^2(1-\alpha)\pi^2}{\alpha n^3} + O_\alpha\left(\frac{j^4}{n^4}\right). \tag{64}$$

Proof

First, we use the following Maclaurin's expansions of η_α and η'_α:

$$\eta_\alpha(x) = \pi - \frac{1-\alpha}{\alpha}x + O_\alpha(x^3), \qquad \eta'_\alpha(x) = -\frac{1-\alpha}{\alpha} + O_\alpha(x^2).$$

Hence, by Proposition 30,

$$\vartheta_{\alpha,n,j} = \frac{j\pi}{n} - \frac{(1-\alpha)j\pi}{\alpha n^2} + O_\alpha\left(\frac{j}{n^3}\right) + O_\alpha\left(\frac{j^3}{n^4}\right).$$

We substitute this expansion into $g(x) = x^2 + O(x^4)$ and obtain (64). □

In particular, (64) can be applied when j is fixed and n tends to ∞. In this situation, (64) provides a better error estimate than the asymptotic formula in Theorem 4.

10. Norms of the eigenvectors

In this section we prove Theorem 5 about the eigenvectors of $L_{\alpha,n,j}$. We suppose that $\alpha \in \mathbb{C}$, $0 < \mathrm{Re}(\alpha) < 1$. Formula (17) follows from Proposition 8. We divide the rest of the proof into three lemmas. Lemma 33, Lemma 34 provide exact formulas (66) and (69) for $\|v_{\alpha,n,j}\|^2$, where j is odd ($j \geq 3$) and even, respectively. In Lemma 35 we prove that for every fixed α and j even, the second term of (69) (which does not contain the factor n) is uniformly bounded with respect to n and j.

In this section we use the following elementary trigonometric identity:

$$\sum_{k=1}^{n}\cos(2kx+y)=\frac{\sin(nx)\cos((n+1)x+y)}{\sin x}. \tag{65}$$

Recall that $v_{\alpha,n,j}$ is the vector with components (17).

Lemma 33

Let $n\geq 3$ and j be odd, $3\leq j\leq n$. Then

$$\|v_{\alpha,n,j}\|_2=|1-\alpha|\sqrt{\tfrac{n}{2}\lambda_{\alpha,n,j}}. \tag{66}$$

Proof

By Theorem 1, it follows that $\vartheta_{\alpha,n,j}=(j-1)\pi/n$ and

$$\sin(n\vartheta_{\alpha,n,j})=0,\qquad \cos(n\vartheta_{\alpha,n,j})=1,\qquad \sin((n-k)\vartheta_{\alpha,n,j})=-\sin(k\vartheta_{\alpha,n,j}).$$

Hence

$$v_{\alpha,n,j,k}=(1-\overline{\alpha})\left(\sin(k\vartheta_{\alpha,n,j})-\sin((k-1)\vartheta_{\alpha,n,j})\right)$$
$$=2(1-\overline{\alpha})\sin\frac{\vartheta_{\alpha,n,j}}{2}\cos\frac{(2k-1)\vartheta_{\alpha,n,j}}{2}.$$

Therefore

$$|v_{\alpha,n,j,k}|^2=4|1-\alpha|^2\sin^2\frac{\vartheta_{\alpha,n,j}}{2}\cos^2\frac{(2k-1)\vartheta_{\alpha,n,j}}{2}$$
$$=g(\vartheta_{\alpha,n,j})|1-\alpha|^2\left(\frac{1+\cos((2k-1)\vartheta_{\alpha,n,j})}{2}\right). \tag{67}$$

Now we sum over k and apply (65):

$$\|v_{\alpha,n,j}\|_2^2=\frac{1}{2}g(\vartheta_{\alpha,n,j})|1-\alpha|^2\left(n+\frac{\sin(2n\vartheta_{\alpha,n,j})}{2\sin\vartheta_{\alpha,n,j}}\right).$$

This implies (66) since $\sin(2n\vartheta_{\alpha,n,j})=\sin(2(j-1)\pi)=0$. □

For every $x\in[0,\pi]$, we define

$$\xi_\alpha(x):=\frac{|1-\alpha|^2}{2}g(x)\cos(\eta_\alpha(x))+\frac{|\alpha|^2}{2}g(\eta_\alpha(x))\cos(x)$$
$$+\frac{\operatorname{Re}(\alpha)-|\alpha|^2}{2}\left(g(x)+g(x+\eta_\alpha(x))-g(\eta_\alpha(x))\right)-2|\alpha|^2\cos(x). \tag{68}$$

Lemma 34

Let $n\geq 3$ and j be even, $2\leq j\leq n$. Then

$$\|v_{\alpha,n,j}\|_2^2=n\nu_\alpha(\vartheta_{\alpha,n,j})+\frac{\sin(\eta_\alpha(\vartheta_{\alpha,n,j}))}{\sin(\vartheta_{\alpha,n,j})}\xi_\alpha(\vartheta_{\alpha,n,j}). \tag{69}$$

Proof

By Theorem 2, $\vartheta_{\alpha,n,j}=(j-1)\pi/n+\eta_\alpha(\vartheta_{\alpha,n,j})/n$. Then

$$\sin(n\vartheta_{\alpha,n,j})=-\sin(\eta_\alpha(\vartheta_{\alpha,n,j})),\qquad \cos(n\vartheta_{\alpha,n,j})=-\cos(\eta_\alpha(\vartheta_{\alpha,n,j})),$$

$$\sin((n-k)\vartheta_{\alpha,n,j})=\sin(k\vartheta_{\alpha,n,j}-\eta_\alpha(\vartheta_{\alpha,n,j})).$$

So, (17) transforms into

$$v_{\alpha,n,j,k} = 2(1-\overline{\alpha})\sin\frac{\vartheta_{\alpha,n,j}}{2}\cos\frac{(2k-1)\vartheta_{\alpha,n,j}}{2} \\ +2\overline{\alpha}\cos\frac{\eta_\alpha(\vartheta_{\alpha,n,j})}{2}\sin\frac{2k\vartheta_{\alpha,n,j}-\eta_\alpha(\vartheta_{\alpha,n,j})}{2}.$$

Then $|v_{\alpha,n,j,k}|^2$ can be written as a sum of three terms:

$$|v_{\alpha,n,j,k}|^2 = \frac{|1-\alpha|^2}{2}g(\vartheta_{\alpha,n,j})(1+\cos((2k-1)\vartheta_{\alpha,n,j})) \\ +\frac{|\alpha|^2}{2}\cdot 4\cos^2\frac{\eta_\alpha(\vartheta_{\alpha,n,j})}{2}(1-\cos(2k\vartheta_{\alpha,n,j}-\eta_\alpha(\vartheta_{\alpha,n,j}))) \\ +4\left(|\alpha|^2-\operatorname{Re}(\alpha)\right)\sin\frac{\vartheta_{\alpha,n,j}}{2}\cos\frac{\eta_\alpha(\vartheta_{\alpha,n,j})}{2}\times \\ \times\left(\sin\frac{\eta_\alpha(\vartheta_{\alpha,n,j})-\vartheta_{\alpha,n,j}}{2}-\sin\left(2k\vartheta_{\alpha,n,j}-\frac{\eta_\alpha(\vartheta_{\alpha,n,j})+\vartheta_{\alpha,n,j}}{2}\right)\right). \tag{70}$$

Now we compute $\sum_{k=1}^n |v_{\alpha,n,j,k}|^2$ working separately with each of the three terms from (70). The sums involving $k\vartheta_{\alpha,n,j}$ are transformed by (65):

$$\sum_{k=1}^n \cos((2k-1)\vartheta_{\alpha,n,j}) = \frac{\sin(\eta_\alpha(\vartheta_{\alpha,n,j}))\cos(\eta_\alpha(\vartheta_{\alpha,n,j}))}{\sin(\vartheta_{\alpha,n,j})}, \tag{71}$$

$$\sum_{k=1}^n \cos(2k\vartheta_{\alpha,n,j}-\eta_\alpha(\vartheta_{\alpha,n,j})) = \frac{\sin(\eta_\alpha(\vartheta_{\alpha,n,j}))\cos(\vartheta_{\alpha,n,j})}{\sin(\vartheta_{\alpha,n,j})}, \tag{72}$$

$$\sum_{k=1}^n \sin\left(2k\vartheta_{\alpha,n,j}-\frac{\eta_\alpha(\vartheta_{\alpha,n,j})+\vartheta_{\alpha,n,j}}{2}\right) = \frac{\sin(\eta_\alpha(\vartheta_{\alpha,n,j}))\sin\frac{\eta_\alpha(\vartheta_{\alpha,n,j})+\vartheta_{\alpha,n,j}}{2}}{\sin(\vartheta_{\alpha,n,j})}. \tag{73}$$

After some elementary simplifications we obtain (69). □

In the next lemma we prove that the second term in (69) is uniformly bounded with respect to *n* and *j*.

Lemma 35

There exists $C_\alpha > 0$, depending only on α, such that for every $n \geq 3$ and every j even, $2 \leq j \leq n$,

$$\left|\frac{\sin(\eta_\alpha(\vartheta_{\alpha,n,j}))}{\sin(\vartheta_{\alpha,n,j})}\xi_\alpha(\vartheta_{\alpha,n,j})\right| \leq C_\alpha. \tag{74}$$

Proof

Obviously, ξ_α is a bounded function on $[0,\pi]$. By a simple application of l'Hôpital's rule, the quotient $\sin(\eta_\alpha(x))/\sin(x)$ has finite limits at 0 and π, hence it is bounded on $[0,\pi]$. This implies (74). □

Proof of Theorem 5

It is a well-known basic fact in the theory of laplacian matrices that the vector $[1,\ldots,1]^\top$ is an eigenvector associated to the eigenvalue $\lambda = 0$. From Proposition 8 we obtain (17). In Lemma 33, (18) has been proved. From Lemma 34, Lemma 35 we obtain (19). □

11. Numerical experiments

With the help of Sagemath, we have verified numerically (for many values of parameters) the representations (31), (32), (34) for the characteristic polynomial, the equivalence of the formulas (7), (38), (39), (40) for η_α, expressions (66), (69) for the norms of the eigenvectors, and some other exact formulas of this paper.

The following web page (written in JavaScript and SVG) contains interactive analogs of Fig. 3, Fig. 5, where the user can choose the values of α and n.
https://www.egormaximenko.com/plots/laplacian_of_cycle_eig.html

We introduce the following notation for different approximations of the eigenvalues.

- $\lambda_{\alpha,n,j}^{\mathrm{gen}}$ are the eigenvalues computed in Sagemath by general algorithms, with double-precision arithmetic.
- $\lambda_{\alpha,n,j}^{\mathrm{N}} := g(\vartheta_{\alpha,n,j}^{\mathrm{N}})$, where $\vartheta_{\alpha,n,j}^{\mathrm{N}}$ is the numerical solution of the equation $h_{\alpha,n,j}(x) = 0$ by Newton's method, see Theorem 3. We use $d_{n,j}$ as the initial approximation. These computations are performed in the high-precision arithmetic with 3322 binary digits (≈1000 decimal digits).
- Using $\vartheta_{\alpha,n,j}^{\mathrm{N}}$ we compute $v_{\alpha,n,j}$ by (17).
- $\lambda_{\alpha,n,j}^{\mathrm{bisec}}$ is similar to $\lambda_{\alpha,n,j}^{\mathrm{N}}$, but now we solve the equation $h_{\alpha,n,j}(x) = 0$ by the bisection method, see Proposition 18.
- $\lambda_{\alpha,n,j}^{\mathrm{fp}}$ is computed similarly to $\lambda_{\alpha,n,j}^{\mathrm{N}}$, but solving the main equation by the fixed point iteration, see Proposition 22.
- $\lambda_{\alpha,n,j}^{\mathrm{N},2}$ is computed similarly to $\lambda_{\alpha,n,j}^{\mathrm{N}}$, but using only two iterations of Newton's method.
- $\lambda_{\alpha,n,j}^{\mathrm{asympt}}$ is the approximation given by (14).

We have constructed a large series of examples including all rational values α in $(0, 1)$ with denominators ≤10 and all n with $3 \le n \le 256$. In all these examples, we have obtained

$$\max_{1\le j\le n} \|L_{\alpha,n} v_{\alpha,n,j} - \lambda_{\alpha,n,j}^{\mathrm{N}} v_{\alpha,n,j}\|_2 < 10^{-996}, \qquad \max_{1\le j\le n} |\lambda_{\alpha,n,j}^{\mathrm{gen}} - \lambda_{\alpha,n,j}^{\mathrm{N}}| < 10^{-13}.$$

Moreover, in all examples

$$\max_{1\le j\le n} |\lambda_{\alpha,n,j}^{\mathrm{N}} - \lambda_{\alpha,n,j}^{\mathrm{bisec}}| < 10^{-998},$$

and for $n > \mathscr{K}_1(\alpha)$,

$$\max_{1\le j\le n} |\lambda_{\alpha,n,j}^{\mathrm{fp}} - \lambda_{\alpha,n,j}^{\mathrm{N}}| < 10^{-998}.$$

For Theorem 4, we have computed the errors

$$R_{\alpha,n,j}^{\mathrm{asympt}} := \lambda_{\alpha,n,j}^{\mathrm{asympt}} - \lambda_{\alpha,n,j}^{\mathrm{N}}$$

and their maximums $\|R_{\alpha,n}^{\mathrm{asympt}}\|_\infty = \max_{1\le j\le n} |R_{\alpha,n,j}^{\mathrm{asympt}}|$. Table 1 shows that these errors indeed can be bounded by $O_\alpha(1/n^3)$.

Table 1. Values of $\|R_{\alpha,n}^{\mathrm{asympt}}\|_\infty$ and $n^3\|R_{\alpha,n}^{\mathrm{asympt}}\|_\infty$ for some α and n.

	$\alpha = 1/3$	
n	$\|R_{\alpha,n}^{\mathrm{asympt}}\|_\infty$	$n^3\|R_{\alpha,n}^{\mathrm{asympt}}\|_\infty$
256	2.28×10^{-6}	38.24
512	2.90×10^{-7}	38.86
1024	3.65×10^{-8}	39.17
2048	4.58×10^{-9}	39.32

	$\alpha = 1/3$	
n	$\|R^{\text{asympt}}_{\alpha,n}\|_\infty$	$n^3\|R^{\text{asympt}}_{\alpha,n}\|_\infty$
4096	5.73×10^{-10}	39.40
8192	7.17×10^{-11}	39.44

	$\alpha = 4/5$	
n	$\|R^{\text{asympt}}_{\alpha,n}\|_\infty$	$n^3\|R^{\text{asympt}}_{\alpha,n}\|_\infty$
256	6.90×10^{-7}	11.58
512	8.66×10^{-8}	11.62
1024	1.08×10^{-8}	11.63
2048	1.36×10^{-9}	11.64
4096	1.69×10^{-10}	11.64
8192	2.12×10^{-11}	11.64

Let $R^{\mathrm{N},2}_{\alpha,n,j} := \lambda^{\mathrm{N},2}_{\alpha,n,j} - \lambda^{\mathrm{N}}_{\alpha,n,j}$ and $\|R^{\mathrm{N},2}_{\alpha,n}\|_\infty = \max_{1\le j\le n}|R^{\mathrm{N},2}_{\alpha,n,j}|$. Table 2 shows that these errors behave as $O_\alpha(1/n^7)$.

Table 2. Values of $\|R^{\mathrm{N},2}_{\alpha,n}\|_\infty$ and $n^7\|R^{\mathrm{N},2}_{\alpha,n}\|_\infty$ for some α and n.

	$\alpha = 1/3$	
n	$\|R^{\mathrm{N},2}_{\alpha,n}\|_\infty$	$n^7\|R^{\mathrm{N},2}_{\alpha,n}\|_\infty$
256	4.13×10^{-17}	2.97
512	3.26×10^{-19}	3.01
1024	2.57×10^{-21}	3.03
2048	2.01×10^{-23}	3.04
4096	1.57×10^{-25}	3.04
8192	1.23×10^{-27}	3.05

	$\alpha = 4/5$	
n	$\|R^{\mathrm{N},2}_{\alpha,n}\|_\infty$	$n^7\|R^{\mathrm{N},2}_{\alpha,n}\|_\infty$
256	6.30×10^{-16}	45.41
512	5.02×10^{-18}	46.33
1024	3.96×10^{-20}	46.80
2048	3.11×10^{-22}	47.04

	$\alpha = 4/5$	
n	$\|R_{\alpha,n}^{\mathrm{N},2}\|_\infty$	$n^7\|R_{\alpha,n}^{\mathrm{N},2}\|_\infty$
4096	2.44×10^{-24}	47.16
8192	1.91×10^{-26}	47.22

We have done similar tests for many other values of α and n. Numerical experiments show that $n^3\|R_{\alpha,n}^{\mathrm{asympt}}\|_\infty$ and $n^7\|R_{\alpha,n}^{\mathrm{N}}\|_\infty$ are bounded by some numbers depending on α, and that numbers grow as α tends to 0 or 1.

Let $R_{\alpha,n,j}^{\mathrm{asympt},2} := \lambda_{\alpha,n,j}^{\mathrm{N}} - (\frac{j^2\pi^2}{n^2} - \frac{2(1-\alpha)j^2\pi^2}{\alpha n^3})$. Table 3 shows that these errors behave as $O_\alpha(j^4/n^4)$.

Table 3. Values of $(n^4/j^4)|R_{\alpha,n,j}^{\mathrm{asympt},2}|$ for $\alpha=1/3$, and some n and even j.

	$\alpha = 1/3$		
n	$(n^4/2^4)\|R_{\alpha,n,2}^{\mathrm{asympt},2}\|$	$(n^4/4^4)\|R_{\alpha,n,4}^{\mathrm{asympt},2}\|$	$(n^4/6^4)\|R_{\alpha,n,6}^{\mathrm{asympt},2}\|$
256	21.80	0.18	4.25
512	21.65	0.44	4.53
1024	21.57	0.58	4.67
2048	21.53	0.65	4.75
4096	21.51	0.68	4.79
8192	21.50	0.70	4.81

Declaration of Competing Interest

The authors declare that they have no competing interest.

Acknowledgement

We are grateful to the referee for useful corrections.

Recommended articles

Data availability

No data was used for the research described in the article.

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1 The research of the first author has been supported by CONACYT (Mexico) project “Ciencia de Frontera” FORDECYT-PRONACES/61517/2020 and by Regional Mathematical Center of the Southern Federal University with the support of the Ministry of Science and Higher Education of Russia, Agreement 075-02-2021-1386.

2 The research of the second author has been supported by CONACYT (Mexico) project “Ciencia de Frontera” FORDECYT-PRONACES/61517/2020 and IPN-SIP project 20220734 (Instituto Politécnico Nacional, Mexico).

3 The research of the third author has been supported by CONACYT (Mexico) PhD scholarship.