Applications of Blaschke Products to the Spectral Theory of Toeplitz Operators

Sergei Grudsky and Eugene Shargorodsky

Abstract The paper is a survey of some applications of Blaschke products to the spectral theory of Toeplitz operators. Topics discussed include Toeplitz operators with bounded measurable symbols, factorisation with an infinite index, compositions with Blaschke products, representation of functions with a given asymptotic behaviour of the argument in a neighbourhood of a discontinuity in the form of a composition of a continuous function with a Blaschke product, and applications to the KdV equation.

Key words: Toeplitz operators, spectral theory, discontinuous symbols, Blaschke products.

Subject Classifications: Primary 47B35, 30J10; Secondary 47A10, 30H10

1 Introduction

Let $\mathbb{T} = \{\zeta \in \mathbb{C} : \ |\zeta| = 1\}$ be the unit circle and let $H^p(\mathbb{T})$, $1 \le p \le \infty$ denote the Hardy space, that is $H^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : \ f_n = 0 \text{ for } n < 0\}$, where f_n is the nth Fourier coefficient of f. Let $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ denote the Toeplitz operator generated by a function $a \in L^\infty(\mathbb{T})$, i.e. $T(a)f = P(af)$, $f \in H^p(\mathbb{T})$, where P is the Riesz projection:

$$Pg(\zeta) = \frac{1}{2}\, g(\zeta) + \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(w)}{w - \zeta}\, dw, \quad \zeta \in \mathbb{T}. \tag{1}$$

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$P : L^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ is a bounded projection and

$$P\left(\sum_{n=-\infty}^{+\infty} g_n \zeta^n\right) = \sum_{n=0}^{+\infty} g_n \zeta^n.$$

Toeplitz operators on the real line are defined similarly: let

$$Pf(x) = \frac{1}{2}\, f(x) + \frac{1}{2\pi i} \int\limits_{\mathbb{R}} \frac{f(\tau)}{\tau - x}\, d\tau, \quad x \in \mathbb{R}. \tag{2}$$

Then $P : L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$ is a bounded projection and its range $H^p(\mathbb{R}) := PL^p(\mathbb{R})$ is the Hardy space corresponding to the upper half plane. The Toeplitz operator generated by a function (symbol) $a \in L^\infty(\mathbb{R})$ is defined as follows

$$T(a)f := P(af), \quad T(a) : H^p(\mathbb{R}) \to H^p(\mathbb{R}).$$

Linear fractional transformations usually allow one to switch between Toeplitz operators on $\mathbb{R}$ and those on $\mathbb{T}$ without difficulty. Most of the present paper deals with the case of $\mathbb{T}$, although we pass to Toeplitz operators on $\mathbb{R}$ when discussing symbols with discontinuities of the (semi-)almost periodic type.

Toeplitz operators are closely related to the Riemann-Hilbert problem. They represent a universal and a most powerful tool that has been applied to a wide variety of problems in elasticity theory, fluid dynamics, physics, geometry, combinatorics, integrable systems, orthogonal polynomials, random matrices, probability and stochastic processes, information and control theory, and in many other fields. Toeplitz operators constitute one of the most important classes of non-self-adjoint operators with a very rich spectral theory, which utilizes methods of operator theory, function theory and the theory of Banach algebras. Their spectral properties are well understood in the case of piece-wise continuous, almost periodic or semi-almost periodic symbols (see the next section for more information and references). Unfortunately much less is known about properties of Toeplitz operators with general bounded measurable symbols.

The aim of the present survey is to describe an approach to the study of spectral properties of Toeplitz operators with symbols having "bad" discontinuities. This approach is based on a generalisation of the Wiener-Hopf factorisation that involves inner functions (Section 4) and on results on representation of functions with a given asymptotic behaviour of the argument in a neighbourhood of the discontinuity in the form of a Blaschke product or, more generally, in the form of a composition of a continuous function with a Blaschke product (Section 5). When dealing with compositions involving Blaschke products in the context of Toeplitz operators, one needs to study compositions of Muckenhoupt weights with Blaschke products. The corresponding results are described in Section 3. Section 2 is a brief introduction to the spectral theory of Toeplitz operators. Section 6 is devoted to applications of Blaschke products to the KdV equation. The final Section 7 contains a list of some open problems.

In order to keep the presentation simple, we do not consider Toeplitz operators on weighted Hardy spaces and block Toeplitz operators, i.e. Toeplitz operators with matrix symbols ($a \in L^\infty_{N\times N}$).

2 Spectra of Toeplitz Operators

A bounded linear operator A on a Banach space X is said to be normally solvable if its range $\operatorname{Ran} A$ is closed. We put $\operatorname{Ker} A = \{f \in X : Af = 0\}$ and $\operatorname{Coker} A := X/\operatorname{Ran} A$. If A is normally solvable and $\dim \operatorname{Ker} A < \infty$, then A is called a Φ_+-operator. If $\dim \operatorname{Coker} A < \infty$, then A is normally solvable and is called a Φ_--operator. A Fredholm operator is an operator that is both Φ_- and Φ_+. The index of a Fredholm operator A is the integer $\operatorname{Ind} A := \dim \operatorname{Ker} A - \dim \operatorname{Coker} A$. The operator A is right (left) invertible if there is a bounded linear operator B on X such that $AB = I$ ($BA = I$), where I is identity operator on X, and the operator A is invertible if there is a bounded operator B on X such that $AB = BA = I$. It is easy to see that if A is left (right) invertible, then A is a $\Phi_+(\Phi_-)$-operator.

The spectrum and the essential spectrum of A are defined as follows:

$$\operatorname{Spec}(A) := \{\lambda \in \mathbb{C} : \ A - \lambda I \ \text{ is not invertible}\},$$
$$\operatorname{Spec}_{\mathrm{e}}(A) := \{\lambda \in \mathbb{C} : \ A - \lambda I \ \text{ is not Fredholm}\}.$$

For any algebra $\mathfrak{A}$, we denote by $G\mathfrak{A}$ the group of invertible elements of $\mathfrak{A}$.

Theorem 1. ([46]) *The spectrum of $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is connected.*

Theorem 2. ([9], [42], see also [17, Ch. 7, Theorem 5.1] or [5, Theorem 2.38]) *Let $a \in L^\infty(\mathbb{T})$, $a \neq 0$. Then $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ has a trivial kernel or a dense range.*

This theorem implies that a nonzero Toeplitz operator $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is normally solvable if and only if it is Φ_- or Φ_+.

Theorem 3. ([22], [42], see also [17, Ch. 7, Theorem 4.1] or [5, Theorem 2.30]) *Let $a \in L^\infty(\mathbb{T})$, $a \neq 0$. If $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is normally solvable then $a \in GL^\infty(\mathbb{T})$, i.e.*

$$\operatorname{ess}\inf_{t\in\mathbb{T}} |a(t)| > 0.$$

Theorem 4. ([10], [44], see also [5, Proposition 2.32]) *Let $a \in L^\infty(\mathbb{T})$. Then $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible (Fredholm, Φ_- or Φ_+) if and only if $a \in GL^\infty(\mathbb{T})$ and $T(a/|a|) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible (Fredholm, Φ_- or Φ_+ respectively). Moreover, if $a \in GL^\infty(\mathbb{T})$, then*

$$\dim \operatorname{Ker} T(a) = \dim \operatorname{Ker} T(a/|a|), \quad \dim \operatorname{Coker} T(a) = \dim \operatorname{Coker} T(a/|a|).$$

Let $C(\mathbb{T})$ be the space of all continuous functions on the unit circle $\mathbb{T}$. Suppose $b \in C(\mathbb{T})$ and $b(t) \neq 0$, $\forall t \in \mathbb{T}$. Then the winding number of b is defined as follows

$$\operatorname{wind} b := \frac{1}{2\pi} [\arg b]_{\mathbb{T}},$$

where $[\arg b]_{\mathbb{T}}$ denotes the total increment of $\arg b(t)$ as the variable t travels around $\mathbb{T}$ in the counterclockwise direction.

Theorem 5. ([15], see also [17, Ch. 3, Theorem 7.1] or [5, Theorem 2.42]) *Let* $a \in C(\mathbb{T})$. *Then* $\operatorname{Spec}_{\mathrm{e}}(T(a)) = a(\mathbb{T})$ *and*

$$\operatorname{Ind}(T(a) - \lambda I) = -\operatorname{wind}(a - \lambda), \quad \forall \lambda \in \mathbb{C} \setminus a(\mathbb{T}).$$

Theorem 6. ([16], see also [17, Ch. 9, Theorem 3.1] or [5, Proposition 5.39]) *Let* $a \in L^\infty(\mathbb{T})$ *be piecewise continuous and let*

$$\operatorname{Arc}_p(a; t) := \left\{ \zeta \in \mathbb{C} \,\middle|\, \arg \frac{a(t-0) - \zeta}{a(t+0) - \zeta} = \frac{2\pi}{p} \right\}$$

if $a(t-0) \neq a(t+0)$. *Then*

$$\operatorname{Spec}_{\mathrm{e}}(T(a)) = \left(\bigcup_{t \in \mathbb{T}} \{a(t \pm 0)\} \right) \bigcup \left(\bigcup_{a(t-0) \neq a(t+0)} \operatorname{Arc}_p(a; t) \right).$$

Let $H^\infty(\mathbb{T}) + C(\mathbb{T})$ be the Banach algebra of all functions of the form $h + f$ with $h \in H^\infty(\mathbb{T})$ and $f \in C(\mathbb{T})$ (see [34, 35]). An element a is invertible in $H^\infty(\mathbb{T}) + C(\mathbb{T})$ if and only if its harmonic extension to the unit disk is bounded away from zero in some annulus $1 - \delta < |z| < 1$ ([11], [12, 7.36], see also [5, Theorem 2.62]).

Theorem 7. ([11], [12, 7.36], see also [5, Theorem 2.65] and [13, Theorem 2.7]) *Suppose* $a \in H^\infty(\mathbb{T}) + C(\mathbb{T})$ *and* $\operatorname{ess\,inf}_{t \in \mathbb{T}} |a(t)| > 0$.

(1) $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ *is Fredholm if and only if* $1/a \in H^\infty(\mathbb{T}) + C(\mathbb{T})$, *in which case* $\operatorname{Ind}(T(a)) = -\operatorname{wind}(a_r)$, *where* $r \in (0,1)$ *is sufficiently close to* 1, $a_r\left(e^{i\theta}\right) := \widehat{a}\left(re^{i\theta}\right)$ *and* $\widehat{a}$ *is the harmonic extension of* a *to the unit disk.*
(2) *If* $1/a \notin H^\infty(\mathbb{T}) + C(\mathbb{T})$, *then* $T(a)$ *is left invertible and* $T(1/a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ *is its left inverse.*

A number $c \in \mathbb{C}$ is called a *(left, right) cluster value* of a measurable function $a : \mathbb{T} \to \mathbb{C}$ at a point $t \in \mathbb{T}$ if $1/(a-c) \notin L^\infty(W)$ for every neighbourhood (left semi-neighbourhood or right semi-neighbourhood respectively) $W \subset \mathbb{T}$ of t. Cluster values are invariant under changes of the function on measure zero sets. We denote the set of all left (right) cluster values of a at t by $a(t-0)$

(by $a(t+0)$), and use also the following notation $a(t) = a(t-0) \cup a(t+0)$, $a(\mathbb{T}) = \cup_{t\in\mathbb{T}} a(t)$. It is easy to see that $a(t-0)$, $a(t+0)$, $a(t)$ and $a(\mathbb{T})$ are closed sets. Hence they are all compact if $a \in L^\infty(\mathbb{T})$.

It follows from Theorem 3 that

$$a(\mathbb{T}) \subseteq \mathrm{Spec}_{\mathrm{e}}(T(a)). \tag{3}$$

Suppose that for each $t \in \mathbb{T}$ the set $a(t)$ consists of two points

$$a_1(t), a_2(t) \in \mathbb{C}$$

(which may coincide). We say that $t \in \mathbb{T}_{\mathrm{I}}$ if $a_1(t) \neq a_2(t)$ and each of the sets $a(t-0)$ and $a(t+0)$ consists of one point, i.e. if a has a left and a right limits at t and they do not coincide. We say that $t \in \mathbb{T}_{\mathrm{II}}$ if at least one of the sets $a(t-0)$, $a(t+0)$ consists of two points, i.e. if a does not have a left or a right limit at t.

Let

$$\mathcal{R}_p(a;t) := \left\{ \zeta \in \mathbb{C} \,\middle|\, \frac{2\pi}{\max\{p,p'\}} \le \arg\frac{a_1(t)-\zeta}{a_2(t)-\zeta} \le \frac{2\pi}{\min\{p,p'\}} \right\}, \tag{4}$$

where $p' = p/(p-1)$.

Theorem 8. ([7, 8, 43], see also [5, 5.50–5.58]) *Suppose $a \in L^\infty(\mathbb{T})$ and for each $t \in \mathbb{T}$ the set $a(t)$ consists of at most two points. Then*

$$\mathrm{Spec}_{\mathrm{e}}(T(a)) = a(\mathbb{T}) \bigcup \left(\bigcup_{t\in\mathbb{T}_{\mathrm{I}}} Arc_p(a;t) \right) \bigcup \left(\bigcup_{t\in\mathbb{T}_{\mathrm{II}}} \mathcal{R}_p(a;t) \right).$$

A complete description of the (essential) spectrum of $T(a)$ in terms of $a(t \pm 0)$, $t \in \mathbb{T}$ is no longer possible if $a(t)$ is allowed to contain more than two points (see [5, 4.71–4.78] and [38]). We return to this topic in Section 4. Here, we continue with a general result on factorisation.

Definition 1. Let $1 < p < \infty$. We say that a function $a \in GL^\infty(\mathbb{T})$ admits a p-factorisation if it can be represented in the form

$$a(t) = a_-(t) t^\kappa a_+(t), \quad t \in \mathbb{T}, \tag{5}$$

where κ is an integer, called the index of factorisation, and the functions $a^\pm$ satisfy the following conditions:

(1) $\overline{a_-} \in H^p(\mathbb{T})$, $\overline{a_-^{-1}} \in H^{p'}(\mathbb{T})$, $a_+ \in H^{p'}(\mathbb{T})$, $a_+^{-1} \in H^p(\mathbb{T})$, $p' = p/(p-1)$;
(2) the operator $(1/a_+)Pa_+I$ is bounded on $L^p(\mathbb{T})$.

It is not difficult to see that a p-factorisation is unique up to a constant factor. The set of all functions $a \in GL^\infty(\mathbb{T})$ that admit a p-factorisation will be denoted by $\mathrm{fact}(p)$.

Theorem 9. ([42, 44, 45], see also [17, Ch. 8, Theorems 4.1 and 4.2] or [5, Theorem 5.5]) *Let $a \in GL^\infty(\mathbb{T})$. The Toeplitz operator $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ is Fredholm if and only if $a \in \mathrm{fact}(p)$. If representation* (5) *holds, then* $\mathrm{Ind}\, T(a) = -\kappa$*, and for $\kappa = 0$ ($\kappa > 0$ or $\kappa < 0$) the operator $T(a)$ is invertible (left invertible or right invertible respectively); moreover,*

$$[T(a)]^{-1} = P \frac{1}{t^\kappa a_+} P \frac{1}{a_-} I \tag{6}$$

is the corresponding inverse operator. Further, for $\kappa < 0$ we have

$$\dim \mathrm{Ker}\, T(a) = |\kappa| \quad \textit{and} \quad \mathrm{Ker}\, T(a) = \mathrm{span}\left\{ \frac{t^{j-1}}{a^+},\ j = 1, 2, \ldots, |\kappa| \right\}, \tag{7}$$

while for $\kappa > 0$ we have $\dim \mathrm{Coker}\, T(a) = \kappa$*, and $f \in \mathrm{Ran}\, T(a)$ if and only if the following orthogonality conditions are satisfied:*

$$\int_{\mathbb{T}} f(t)\, \frac{1}{t^j a_-(t)}\, dt = 0, \quad j = 1, 2, \ldots, \kappa. \tag{8}$$

It is not always easy to check whether or not $a \in \mathrm{fact}(p)$. The following result describes a rather broad subclass of $\mathrm{fact}(p)$. A function $a \in GL^\infty(\mathbb{T})$ is called *locally p-sectorial* if for every $t \in \mathbb{T}$ there exist an open arc $\ell(t) \subset \mathbb{T}$ containing t and functions $g_\pm^{(t)} \in GH^\infty(\mathbb{T})$ such that

$$\bigcup_{\tau \in \ell(t)} \left(\overline{g_-^{(t)}}\, a g_+^{(t)} \right)(\tau) \subset \left\{ z = re^{i\theta} \in \mathbb{C} :\ r > 0,\ |\theta| < \frac{\pi}{\max\{p, p'\}} \right\}.$$

It is easy to see that $a \in GL^\infty(\mathbb{T})$ is locally p-sectorial if $a(t)$ lies in an open sector with the vertex at the origin and an angular opening not exceeding $2\pi/\max\{p, p'\}$ for every $t \in \mathbb{T}$.

Theorem 10. ([41, 42], see also [17, Ch. 12] or [5, 5.12–5.21]) *Let $a \in GL^\infty(\mathbb{T})$ be locally p-sectorial. Then $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ is Fredholm.*

In the case of the space $L^p(\mathbb{R})$ the notion of a p-factorisation takes the following form. We say that a function $a \in GL^\infty(\mathbb{R})$ admits a *p-factorisation with respect to the real line* $\mathbb{R}$ if it can be represented in the form

$$a(x) = a_-(x) \left(\frac{x - i}{x + i} \right)^\kappa a_+(x), \quad x \in \mathbb{R}, \tag{9}$$

where κ is an integer, called the *index of factorisation*, and the functions $a_\pm$ satisfy the following conditions:

(1) $\dfrac{a_-(x)}{x-i} \in \overline{H^p(\mathbb{R})}, \quad \dfrac{1}{a_-(x)(x-i)} \in \overline{H^{p'}(\mathbb{R})},$

$\dfrac{a_+(x)}{x+i} \in H^{p'}(\mathbb{R}), \quad \dfrac{1}{a_+(x)(x+i)} \in H^p(\mathbb{R}), \quad p' = p/(p-1);$

(2) the operator $(1/a_+)Pa_+I$ is bounded in $L^p(\mathbb{R})$.

The algebra $AP(\mathbb{R})$ of almost periodic functions is defined as the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains the set $\{e_\lambda : \lambda \in \mathbb{R}\}$, where $e_\lambda(x) = e^{i\lambda x}$. We denote by $C(\overline{\mathbb{R}})$ the set of all continuous functions f on $\mathbb{R}$ that have finite limits $f(-\infty)$ and $f(+\infty)$ at $\pm\infty$, and by $C(\dot{\mathbb{R}})$ the subspace of $C(\overline{\mathbb{R}})$ consisting of functions continuous at infinity, i.e. such that $f(-\infty) = f(+\infty)$. Finally, the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains $AP(\mathbb{R}) \cup C(\overline{\mathbb{R}})$ is denoted by $SAP(\mathbb{R})$ and is called the algebra of semi-almost periodic functions. Every function $b \in SAP(\mathbb{R})$ can be represented in the form

$$b(x) = (1 - w(x))b_l(x) + w(x)b_r(x) + c_0(x), \tag{10}$$

where $b_l, b_r \in AP(\mathbb{R})$, $c_0 \in C(\dot{\mathbb{R}})$ with $c_0(\infty) = 0$, and w is a function from $C(\overline{\mathbb{R}})$ such that

$$w(-\infty) = 0 \quad \text{and} \quad w(+\infty) = 1 \tag{11}$$

(see [36]). The functions b_l and b_r are uniquely determined and independent of the choice of w. They are called the left and the right almost periodic representatives of b.

According to H. Bohr's theorem, every function $b \in GAP(\mathbb{R})$ can be written in the form

$$b(x) = e^{i\mu(b)x + c(x)}, \quad x \in \mathbb{R} \tag{12}$$

with $\mu(b) \in \mathbb{R}$ and $c \in AP(\mathbb{R})$. The number $\mu(b)$ is called the mean motion of b and it is given by the formula

$$\mu(b) = \lim_{T\to\infty} \frac{1}{2T} \arg b(x)\Big|_{x=-T}^{T},$$

where $\arg b$ is any continuous branch of the argument of b. If $\mu(b) = 0$, the geometric mean $\lambda(b)$ is defined by

$$\lambda(b) = e^{M(c)}, \tag{13}$$

where $M(c)$ is the mean value of c,

$$M(c) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c(x)dx.$$

Note that the function $c \in AP(\mathbb{R})$ in (12) is unique up to an additive constant in $2\pi i\mathbb{Z}$. Hence definition (13) does not depend on a particular choice of c.

For a $b \in SAP(\mathbb{R})$, set

$$\mu_-(b) := \mu(b_l), \quad \mu_+(b) := \mu(b_r)$$

(see (10)). If $\mu_\pm(b) = 0$, set

$$\lambda_-(b) := \lambda(b_l), \quad \lambda_+(b) := \lambda(b_r).$$

Theorem 11. ([36, 31, 32, 33], see also [13, Theorem 4.24]) *Let $a \in SAP(\mathbb{R})$. If $T(a) : H^p(\mathbb{R}) \to H^p(\mathbb{R})$, $1 < p < \infty$ is normally solvable, then*

$$\inf_{x\in\mathbb{R}} |a(x)| > 0.$$

Suppose this condition is satisfied.

(1) *If $\mu_\pm(a) = 0$, then $T(a)$ is Fredholm if and only if*

$$\frac{1}{2\pi} \arg \frac{\lambda_+(a)}{\lambda_-(a)} - \frac{1}{p} \notin \mathbb{Z}.$$

If this condition is not satisfied, then the $T(a) : H^p(\mathbb{R}) \to H^p(\mathbb{R})$ is not normally solvable.

(2) *If $\mu_\pm(a) \geq 0$ and $\mu_+^2(a) + \mu_-^2(a) \neq 0$, then $T(a)$ is left invertible and* $\dim \operatorname{Coker} T(a) = \infty$.

(3) *If $\mu_\pm(a) \leq 0$ and $\mu_+^2(a) + \mu_-^2(a) \neq 0$, then $T(a)$ is right invertible and* $\dim \operatorname{Ker} T(a) = \infty$.

(4) *If $\mu_+(a)\mu_-(a) < 0$, then $T(a)$ is not normally solvable in any of the spaces $H^p(\mathbb{R})$, $1 < p < \infty$ and* $\dim \operatorname{Ker} T(a) = \dim \operatorname{Coker} T(a) = 0$.

3 Compositions with Blaschke Products and the A_p Condition

The results in Section 2 give an explicit description of the (essential) spectrum of $T(a)$ if $a(t)$ consists of at most two points for every t or if a is semi-almost periodic. Both cases include piecewise continuous symbols treated in Theorem 6. Suppose now $a \in L^\infty(\mathbb{T})$ has a "bad" discontinuity at $t = 1$ or at any other point of $\mathbb{T}$. Then one cannot, in general, tell whether or not $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is Fredholm. A possible way of approaching this problem is to try representing a in the form $a = a_0 \circ v$, where a_0 is a simple, e.g., piecewise continuous function and $v : \mathbb{T} \to \mathbb{T}$ is a suitable measurable transformation. If $v(1) = \mathbb{T}$, then $a = a_0 \circ v \in L^\infty(\mathbb{T})$ has a bad discontinuity at $t = 1$, namely $a(1) = a_0(\mathbb{T})$.

Suppose $T(a_0) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is Fredholm. Then a_0 admits a factorisation of the form (5)

$$a_0(t) = a_-(t)t^\kappa a_+(t), \quad t \in \mathbb{T} \quad (\kappa \in \mathbb{Z})$$

(see Theorem 9). Hence

$$a(t) = a_-(v(t))v^\kappa(t)a_+(v(t)), \quad t \in \mathbb{T}. \tag{14}$$

Since we would like to have

$$\begin{aligned} &\overline{a_- \circ v} \in H^p(\mathbb{T}), \quad \overline{(a_- \circ v)^{-1}} \in H^{p'}(\mathbb{T}), \\ &a_+ \circ v \in H^{p'}(\mathbb{T}), \quad (a_+ \circ v)^{-1} \in H^p(\mathbb{T}), \end{aligned} \tag{15}$$

we need v to have an analytic extension to the unit disk. Given that $|v| = 1$ on $\mathbb{T}$, it is natural to assume that v is a nonconstant inner function. Since $v(1) = \mathbb{T}$, natural choices for v are the singular inner function

$$v(\zeta) = \exp\left(\sigma \frac{\zeta + 1}{\zeta - 1}\right), \quad \sigma = \text{const} > 0 \tag{16}$$

and infinite Blaschke products with zeroes converging to $t = 1$.

Suppose v is an inner function. Then the following variant of Littlewood's subordination principle shows that (15) does indeed hold.

Theorem 12. ([28], [37, Section 1.3] and [13, Theorem 5.5]) *Let v be a nonconstant inner function and let γ_v be defined by*

$$(\gamma_v f)(t) = f(v(t)), \quad t \in \mathbb{T}.$$

(1) *The mapping γ_v is a bounded linear operator on the space $L^p(\mathbb{T})$, $1 \le p < \infty$. The subspace $H^p(\mathbb{T})$ is invariant under γ_v.*
(2) *The mapping γ_v is an automorphism of the algebra $L^\infty(\mathbb{T})$. The subalgebra $H^\infty(\mathbb{T})$ is invariant under γ_v.*
(3) *For any $f \in L^p(\mathbb{T})$, $1 \le p \le \infty$,*

$$\left(\frac{1 - |v(0)|}{1 + |v(0)|}\right)^{1/p} \|f\|_p \le \|\gamma_v f\|_p \le \left(\frac{1 + |v(0)|}{1 - |v(0)|}\right)^{1/p} \|f\|_p. \tag{17}$$

The middle factor in the factorisation (5) is the finite Blaschke product t^κ and the index of the corresponding Toeplitz operator is $-\kappa$. If $\kappa \ne 0$ in (14) and if v is an inner function which is not a finite Blaschke product, then one would expect $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ to be semi-Fredholm with an infinite index. This is indeed the case under natural conditions on the first and the third factors, and the corresponding representation is called a generalised factorisation with an infinite index. The function $a = a_0 \circ v$ is called v-periodic. These notions are discussed in Section 4 (see Theorems 16 and 17).

Finally, we need to find out whether or not the factorisation (14) satisfies condition (2) of Definition 1, i.e. whether or not the operator

$$\frac{1}{a_+ \circ v}\, P(a_+ \circ v)I : L^p(\mathbb{T}) \to L^p(\mathbb{T})$$

is bounded.

Let $\rho : \mathbb{T} \to [0, +\infty]$ be a measurable function. According to the Hunt–Muckenhoupt–Wheeden theorem ([24]), the operator $(1/\rho)P\rho I$ is bounded on $L^p(\mathbb{T})$, $1 < p < \infty$ if and only if ρ satisfies the A_p condition:

$$\sup_I \left(\frac{1}{|I|}\int_I \rho^p(t)|dt|\right)^{\frac{1}{p}} \left(\frac{1}{|I|}\int_I \rho^{-p'}(t)|dt|\right)^{\frac{1}{p'}} = C_p < \infty, \tag{18}$$

where $I \subset \mathbb{T}$ is an arbitrary arc, $|I|$ denotes its length, and $p' = p/(p-1)$.

Hence we arrive at the following question:

$$\text{does } \ \rho \in A_p \ \text{ imply } \ \rho \circ v \in A_p \ \text{ for an arbitrary inner function } \ v? \tag{19}$$

Note by the way that if $v(0) = 0$, then $v : \mathbb{T} \to \mathbb{T}$ is measure preserving, i.e. $\left|v^{-1}(E)\right| = |E|$ for any measurable $E \subset \mathbb{T}$ (see, e.g., [28] or take f equal to the indicator function of E in (17) with $p = 1$).

Using Theorem 9 one can easily show (see [2, Section 1]) that (19) is equivalent to the following question: does the invertibility of $T(b) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ imply that of $T(b \circ v) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$?

The answer is positive in the case $p = 2$ (see, e.g., [2, Theorem 3]). This follows, e.g., from the Helson–Szegö theorem ([23], see also [14, Ch. IV, Theorem 3.4]):

$$\rho \in A_2 \quad \Longleftrightarrow \quad \rho = \exp(f + \widetilde{g}), \quad f, g \in L^\infty(\mathbb{T}, \mathbb{R}), \quad \|g\|_\infty < \pi/4,$$

where $\widetilde{g}$ is the harmonic conjugate of g.

Similarly, a theorem by N.Ya. Krupnik ([25, 26], see also [17, Section 12.5]) says that

$$\begin{aligned} \rho \in A_p \cap A_{p'} \quad \Longleftrightarrow \quad & \rho = \exp(f + \widetilde{g}), \quad f, g \in L^\infty(\mathbb{T}, \mathbb{R}), \\ & \|g\|_\infty < \frac{\pi}{2\max\{p, p'\}}\,, \quad p' = \frac{p}{p-1}\,, \end{aligned}$$

and it is not difficult to show that

$$\rho \in A_p \cap A_{p'} \quad \Longrightarrow \quad \rho \circ v \in A_p \cap A_{p'}$$

(see [2, Theorem 4]).

One can also prove that the reverse of the implication in (19) is true.

Theorem 13. ([2]) *Let $1 < p < \infty$, $p' = p/(p-1)$ and let v be an inner function.*

(1) *Suppose ρ is a weight such that $\rho \in L^p$ and $\rho^{-1} \in L^{p'}$. If $\rho \circ v \in A_p$ then $\rho \in A_p$.*
(2) *Suppose $a \in L^\infty(\mathbb{T})$. If $T(a \circ v) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible then $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is invertible.*

In spite of all the above results, the answer to (19) turns out to be negative.

Let $\{z_k\}_{k=-\infty}^{\infty}$ be a sequence of nonzero points in the open unit disk satisfying

$$\lim_{k\to\pm\infty} z_k = 1 \quad \text{and} \quad \sum_{k=-\infty}^{\infty} (1 - |z_k|) < \infty. \tag{20}$$

The first condition in (20) guarantees that the Blaschke product

$$B(t) := \prod_{k=-\infty}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - t}{1 - \overline{z_k}\, t}, \quad t \in \mathbb{T} \tag{21}$$

extends to an analytic function on $\mathbb{C} \setminus \big(\cup_k \{\overline{z}_k^{-1}\} \cup \{1\}\big)$. In particular, B is continuous on $\mathbb{T} \setminus \{1\}$.

Write $z_k = r_k e^{i\theta_k}$ with $0 < r_k < 1$ and $-\pi < \theta_k \le \pi$. Put

$$\begin{aligned}
\theta_k &:= \begin{cases} (\operatorname{sign} k) e^{-|k|} & \text{for } k \ne 0, \\ -1 & \text{for } k = 0, \end{cases} \\
\varDelta_k &:= \begin{cases} \theta_k - \theta_{k+1} & \text{for } \quad k = 1, 2, 3, \dots, \\ \theta_{k-1} - \theta_k & \text{for } k = 0, -1, -2, \dots, \end{cases} \\
\delta_k &:= \min\left\{ \left(\frac{\varDelta_k}{\log \varDelta_k} \right)^2, \left(\frac{\varDelta_{k-1}}{\log \varDelta_k} \right)^2 \right\},
\end{aligned} \tag{22}$$

choose a number $M > 1$, and define $r_k \in (0,1)$ by

$$r_k := (1 - \delta_k/M)/(1 + \delta_k/M). \tag{23}$$

Theorem 14. ([2]) *Let $p \in (1,2) \cup (2,\infty)$, $1/p + 1/p' = 1$ and let σ be any number in the interval $(1/p', 1/p)$ if $1 < p < 2$ and any number in the interval $(-1/p', -1/p)$ if $2 < p < \infty$. Then*

$$w(t) := |t - 1|^{-\sigma} \tag{24}$$

is a weight in A_p, but if $M > 1$ is sufficiently large and $B_M = B$ is the Blaschke product (21) with the zeroes given by (22)–(23), then

$$w(B_M(t)) = |B_M(t) - 1|^{-\sigma} \tag{25}$$

is not a weight in A_p.

Theorem 14 shows that there exists a Blaschke product for which the implication in (19) does not hold. We now describe a class of Blaschke products for which this implication does hold.

Consider the Blaschke product

$$B\left(e^{i\theta}\right) = \prod_{k=1}^{\infty} \frac{r_k - e^{i\theta}}{1 - r_k e^{i\theta}}\,, \quad \theta \in [-\pi, \pi], \tag{26}$$

where $r_k \in (0, 1)$ and $\sum_{k=1}^{\infty}(1 - r_k) < \infty$.

Theorem 15. ([21]) *Suppose* $r_1 \le r_2 \le \cdots \le r_n \le \cdots,$ *and*

$$\inf_{k\ge 1} \frac{1 - r_{k+1}}{1 - r_k} > 0. \tag{27}$$

If ρ *satisfies the* A_p *condition, then* $\rho \circ B$ *also satisfies the* A_p *condition.*

Corollary. ([21]) *Let* $1 < p < \infty,$ $a \in L^\infty(\mathbb{T})$*, and let a Blaschke product* B *satisfy the conditions of Theorem 15. Then* $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ *is invertible if and only if* $T(a \circ B) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ *is invertible.*

Proof. The invertibility of $T(a \circ B)$ implies that of $T(a)$ according to Theorem 13 (see [2, Theorem 12]). The opposite implication follows from Theorem 15 (see [2, Section 1]).

Theorem 15 and its Corollary remain true if the Blaschke product (26), (27) is substituted with the singular inner function (16) (see [19, 21]).

4 More on the Spectra of Toeplitz Operators

We start with extending Definition 1 and Theorem 9 to the case of $\Phi_\pm$ operators. We say that a function $a \in L^\infty(\mathbb{T})$ *admits a generalised factorisation with an infinite index in the space* $L^p(\mathbb{T})$*,* $1 < p < \infty$ if it admits a representation

$$a = bh \quad \text{or} \quad a = bh^{-1}, \tag{28}$$

where

(1) $b \in \mathrm{fact}(p)$ (see Definition 1);
(2) $h \in H^\infty(\mathbb{T}),\ 1/h \in L^\infty(\mathbb{T})$;
(3) $q/h \notin H^\infty(\mathbb{T})$ for any polynomial q.

In this case, we also say that a *admits an* (h, p)*-factorisation.*

The class of functions admitting a generalized factorisation with an infinite index in $L^p(\mathbb{T})$ will be denoted by $\mathrm{fact}(\infty, p)$.

Let $Q := I - P$, where P is the projection defined by (1).

Theorem 16. ([13, Theorem 2.6]) *Assume* $a \in \mathrm{fact}(\infty, p)$ *and* $\mathrm{ind}\, b = 0$. *If* $a = bh^{-1}$, *then the operator* $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ *is right invertible,* $\dim \mathrm{Ker}\, T(a) = \infty$, *and the operator*

$$[T(a)]^{-1} = \frac{h}{b^+} P \frac{1}{b^-} I, \tag{29}$$

where $b = b^+ b^-$ *is the* p*-factorisation of the function* b, *is a right inverse of* $T(a)$. *For a function* φ *to belong to* $\mathrm{Ker}\, T(a)$ *it is necessary and sufficient that*

$$\varphi = \frac{h}{b^+} Q \frac{b^+}{h} \psi, \quad \textit{where} \quad \psi \in \mathrm{Ker}\, T(h^{-1}). \tag{30}$$

If $a = bh$, *then* $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ *is left invertible,* $\dim \mathrm{Coker}\, T(a) = \infty$, *and the operator*

$$[T(a)]^{-1} = P \frac{1}{b^+ h} P \frac{1}{b^-} I \tag{31}$$

is a left inverse for $T(a)$. *For a function* f *to belong to* $\mathrm{Ran}\, T(a)$ *it is necessary and sufficient that*

$$\int_\Gamma \psi_j(t) f(t) dt = 0, \quad j = 1, 2, \ldots, \tag{32}$$

where

$$\psi_j = \frac{1}{b^-} Q \frac{(t - z_0)^{-j}}{hb^+} \in L_-^{p'}(\mathbb{T}) := QL^{p'}(\mathbb{T}), \quad p' = p/(p-1),$$

and $z_0 \in \mathbb{C}$ *is a fixed point such that* $|z_0| > 1$.

Functions admitting a generalized factorisation with an infinite index often arise as compositions with inner functions (cf. (14)). A function $a \in L^\infty(\mathbb{T})$ is called *u-periodic* if it admits a representation

$$a(t) = g(u(t)), \tag{33}$$

where $g \in L^\infty(\mathbb{T})$ and u is an inner function.

Theorem 17. ([18, Theorem 5.2]) *Let* $g \in C(\mathbb{T})$ *and suppose* $g(t) \neq 0$, $\forall t \in \mathbb{T}$ *and* $\mathrm{wind}\, g = \kappa$. *Then for every* $1 < p < \infty$ *and every inner function* $u \in H^\infty(\mathbb{T})$ *the* u*-periodic function* (33) *admits a* $(u^{|\kappa|}, p)$*-factorisation*

$$a(t) = g_-(u(t)) u^\kappa(t) g_+(u(t)),$$

where $g(t) = g_-(t) t^\kappa g_+(t)$ *is a factorisation of the type* (5). *Moreover, if* g *is a rational function, then*

$$(g_+ \circ u)^{\pm 1} \in H^\infty(\mathbb{T}), \quad (g_- \circ u)^{\pm 1} \in \overline{H^\infty(\mathbb{T})}.$$

Remark 1. Theorem 17 cannot be extended to arbitrary symbols $g \in \mathrm{fact}(p)$ due to the difficulty described by Theorem 14. However, it can be extended to all locally p-sectorial symbols g (see [13, Theorem 5.8]). It also holds for all $g \in \mathrm{fact}(p)$ if one restricts the class of inner functions u to those for which the conclusions of Theorem 15 and its Corollary hold.

Similarly to the situation with Theorem 9, it is not always easy to check whether or not $a \in \mathrm{fact}(\infty, p)$. A broad subclass of $\mathrm{fact}(\infty, p)$ is desribed in Section 5 in terms of the asymptotic behaviour of the argument in a neighbourhood of a discontinuity.

Let us now consider compositions with homeomorphisms $\alpha : \mathbb{R} \to \mathbb{R}$ in the context of Toeplitz operators with (semi-)almost periodic symbols on $\mathbb{R}$. We will confine ourselves to the $H^2(\mathbb{R})$ setting to avoid difficulties related to Theorem 14. We start with a negative result.

Theorem 18. ([3]) *There exist $b \in GAP(\mathbb{R})$ and an orientation preserving homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ such that $T(b) : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is Fredholm while $T(a)$ with $a(x) = b(\alpha(x))$ is not.*

In order to obtain positive results, one needs to restrict the class of homeomorphisms $\alpha : \mathbb{R} \to \mathbb{R}$. Let, similarly to the case of $\mathbb{T}$ considered in Section 2, $H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$ be the Banach algebra of all functions of the form $h + f$ with $h \in H^\infty(\mathbb{R})$ and $f \in C(\dot{\mathbb{R}})$ (see [34, 35]).

Theorem 19. ([3]) *Let $b \in AP(\mathbb{R})$ and suppose*

$$e^{i\lambda\alpha} \in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}}), \quad \forall \lambda > 0. \tag{34}$$

Put $a(x) = b(\alpha(x))$. We then have the following.

(i) If $T(b) : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is invertible, then $T(a)$ is a Φ-operator.
(ii) If $T(b) : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is left invertible, then $T(a)$ is a Φ_+-operator.
(iii) If $T(b) : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is right invertible, then $T(a)$ is a Φ_--operator.

Theorem 36 provides sufficient conditions for (34) to hold (see also Theorem 37).

The following result extends Theorem 19 to semi-almost periodic symbols and it is natural to substitute condition (34) with the following one

$$(1 - w)e^{i\lambda\alpha}, \ we^{i\lambda\alpha} \in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}}) \quad \text{for all} \quad \lambda > 0, \tag{35}$$

where $w \in C(\overline{\mathbb{R}})$ is a fixed function subject to (11).

Theorem 20. ([3]) *Let the homeomorphism α satisfy condition (35) and let $b \in SAP(\mathbb{R})$. Put $a(x) = b(\alpha(x))$. If $T(b) : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is a Φ-operator, then $T(a)$ is also a Φ-operator.*

Let us now return to the comment made after Theorem 8. Consider, for example, $a \in L^\infty(\mathbb{T})$ such that $a(1)$ consists of three points, $a(1\pm 0) = a(\mathbb{T}) = \{c_1, c_2, c_3\} \subset \mathbb{C}$ and the closed triangle $\triangle(c_1, c_2, c_3)$ with the vertices c_1, c_2, c_3 is non-degenerate. Then the (essential) spectrum of $T(a) : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is a connected set which contains $\{c_1, c_2, c_3\}$ and is contained in $\triangle(c_1, c_2, c_3)$ ([12, Theorem 7.45], [22], [6], see also Theorems 1, 3, 10 above). It turns out however that this set is not determined solely by c_1, c_2, c_3. A. Böttcher has constructed examples where the spectrum of $T(a) : H^2(\mathbb{T}) \to H^2(\mathbb{T})$

(i) does not contain any points of the boundary of the triangle $\triangle(c_1, c_2, c_3)$ other than c_1, c_2, c_3;
(ii) contains a side of $\triangle(c_1, c_2, c_3)$ and no other point of the boundary apart from c_1, c_2, c_3;
(iii) coincides with the union of two sides of $\triangle(c_1, c_2, c_3)$;
(iv) coincides with the boundary of $\triangle(c_1, c_2, c_3)$;
(v) coincides with $\triangle(c_1, c_2, c_3)$

(see [5, 4.71–4.78]). These striking examples and the results obtained in [38, 39] imply that if $a(t)$ is not required to contain at most two points for every $t \in \mathbb{T}$, then it is no longer possible to describe the (essential) spectrum of $T(a)$ in terms of the cluster values of a. In other words, it is no longer sufficient to know the values of a, it is rather important to know "how these values are attained" by a. This field seems to be wide open at present.

Since a complete description of the essential spectrum of $T(a)$ in terms of the cluster values of $a \in L^\infty(\mathbb{T})$ is impossible, it is natural to try finding "optimal" sufficient conditions for a point λ to belong to the essential spectrum.

We need the following notation. Let $K \subset \mathbb{C}$ be an arbitrary compact set and $\lambda \in \mathbb{C} \setminus K$. Then the set

$$\sigma(K; \lambda) = \left\{ \frac{w - \lambda}{|w - \lambda|} \;\Big|\; w \in K \right\} \subseteq \mathbb{T}$$

is compact as a continuous image of a compact set. Hence the set $\varDelta_\lambda(K) := \mathbb{T} \setminus \sigma(K; \lambda)$ is open in $\mathbb{T}$. So, $\varDelta_\lambda(K)$ is the union of an at most countable family of open arcs.

We call an open arc of $\mathbb{T}$ p–large if its length is greater than or equal to $2\pi / \max\{p, p'\}$, where $p' = p/(p-1)$, $1 < p < \infty$.

We know that $a(\mathbb{T}) \subseteq \mathrm{Spec_e}(T(a))$ (see (3)). Böttcher's examples mentioned above show that no point from $\mathbb{C} \setminus a(\mathbb{T})$ will always belong to the (essential) spectrum of $T(a) : H^2(\mathbb{T}) \to H^2(\mathbb{T})$, unless $a(\mathbb{T})$ lies on a straight line. The following result shows that the situation is somewhat different for $p \neq 2$.

Theorem 21. ([40]) *Let* $1 < p < \infty$, $a \in L^\infty(\mathbb{T})$, $\lambda \in \mathbb{C} \setminus a(\mathbb{T})$ *and suppose that, for some* $t \in \mathbb{T}$,
(i) $\varDelta_\lambda(a(t-0))$ *(or* $\varDelta_\lambda(a(t+0))$*) contains at least two* p*–large arcs,*

(ii) $\varDelta_\lambda(a(t+0))$ *(or* $\varDelta_\lambda(a(t-0))$ *respectively) contains at least one* p*–large arc.*
Then λ *belongs to the essential spectrum of* $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$*.*

Suppose $a(t)$ consists of two points. Then condition *(ii)* in the above theorem is automatically satisfied, while condition *(i)* means that a does not have a left limit at t (or a right limit at t respectively) and that λ belongs to the set (4). Hence, Theorem 21 is in a sense an extension of Theorem 8.

Condition *(i)* is optimal in the following sense.

Theorem 22. ([39]) *Let* $t \in \mathbb{T}$*,* $K \subset \mathbb{C}$ *be a compact set,* $\lambda \in \mathbb{C} \setminus K$*, and suppose* $\varDelta_\lambda(K)$ *contains at most one* p*–large arc. Then there exists* $a \in L_\infty(\mathbb{T})$ *such that*

$$a(t \pm 0) = a(t) = a(\mathbb{T}) = K$$

and

$$T(a) - \lambda I : H^r(\mathbb{T}) \to H^r(\mathbb{T})$$

is invertible for any $r \in [\min\{p, p'\}, \max\{p, p'\}]$*.*

While condition *(i)* is the main reason why λ belongs to $\mathrm{Spec}_e(T(a))$, the rôle of *(ii)* is to make sure that the behaviour of $a(\tau)$ as τ approaches t from the other side does not counterbalance the effect of *(i)*. It turns out that condition *(ii)* cannot be dropped.

Theorem 23. ([21]) *There exists* $a \in L^\infty(\mathbb{T})$ *such that* $a(1-0) = \{\pm 1\}$*,* $|a| \equiv 1$*,* $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ *is invertible for any* $p \in (1, 2)$*, and* $T(1/a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ *is invertible for any* $p \in (2, +\infty)$*.*

The proof of this theorem relies on the Corollary of Theorem 15 and on Theorem 34.

5 Modelling of Monotone Functions with the Help of Blaschke Products

Suppose $a \in GL^\infty(\mathbb{T})$. Then Theorem 4 allows one to reduce the study of the operator $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ to that of $T(a/|a|) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$. We can therefore assume without loss of generality that $|a| = 1$, i.e. that

$$a(\exp(i\theta)) = \exp(2\pi i f(\theta)), \quad \theta \in (-\pi, \pi], \tag{36}$$

where f is a measurable real-valued function. Suppose a has a discontinuity at $t = 1$. We aim at finding conditions on the behaviour of f in a neighbourhood of $t = 1$ under which a can be represented in terms of Blaschke products in such a way that one can then apply Theorems 16 and 17. Although our motivation comes from the theory of Toeplitz operators, we believe that the results presented in this section may be of some interest in their own right.

Since the discontinuity is at $t = 1$, it is natural to consider Blaschke products with zeroes converging to 1. Our first result is about the argument of such a Blaschke product. Let

$$B\left(e^{i\theta}\right) = \prod_{k=1}^{\infty} \frac{\overline{z_k}}{|z_k|} \, \frac{z_k - e^{i\theta}}{1 - \overline{z_k}\, e^{i\theta}}\,, \quad \theta \in (-\pi, \pi], \tag{37}$$

where $z_k = r_k \exp(i\theta_k) \neq 0,\ \theta_k \in (-\pi, \pi],\ r_k = |z_k| < 1,\ \sum_{k=1}^{\infty}(1 - r_k) < \infty$.

Theorem 24. ([13, Theorem 2.8]) *Suppose B has the form (37) and*

$$\lim_{k\to\infty} z_k = 1.$$

Then one can choose a branch of $\arg B\left(e^{i\tau}\right)$ which is continuous and increasing on $(0, 2\pi)$, and which satisfies the following conditions

$$\lim_{\tau\to 0+0} \arg B\left(e^{i\tau}\right) =: A_+ < 0, \qquad \lim_{\tau\to 2\pi-0} \arg B\left(e^{i\tau}\right) =: A_- > 0.$$

Moreover, at least one of these limits is infinite and

$$\arg B\left(e^{i\theta}\right) = \begin{cases} -2\Big(\sum_{\theta_k \geq \theta} (\pi + \varphi_k(\theta)) + \sum_{\theta_k < \theta} \varphi_k(\theta) \Big), \ \theta \in (0, \pi], \\[2ex] \quad 2\Big(\sum_{\theta_k \leq \theta} (\pi - \varphi_k(\theta)) - \sum_{\theta_k > \theta} \varphi_k(\theta) \Big), \ \theta \in [-\pi, 0), \end{cases} \tag{38}$$

where

$$\varphi_k(\theta) = \arctan\left(\varepsilon_k \cot \frac{\theta - \theta_k}{2}\right), \quad \varepsilon_k = \frac{1 - r_k}{1 + r_k}\,. \tag{39}$$

The next result shows that the argument of a Blaschke product may grow arbitrarily slowly or arbitrarily fast as $t \to 1$ and that the growth on the left from 1 may be different from that on the right.

Theorem 25. ([13, Theorem 2.9]) *Suppose that a real-valued function f is continuous and increasing on $(-\pi, 0)$ and $(0, \pi)$, and that at least one of the limits $\lim_{\theta\to 0\pm 0} f(\theta)$ is infinite. Then there exists a Blaschke product B of the form (37) such that*

$$|\arg B\left(e^{i\theta}\right) - f(\theta)| \leq \text{const}\,, \quad \theta \in (-\pi, \pi) \setminus \{0\}. \tag{40}$$

Theorem 25 allows one to factor out a Blaschke product from the symbol of a Toeplitz operator in such a way that the resulting Toeplitz operator has a symbol with a bounded and continuous argument on $\mathbb{T} \setminus \{1\}$. Unfortunately not much is known about such operators, so the above theorem is not sufficient for our purposes.

Suppose a has the form (36), where the function f is continuous and monotonically increasing on the intervals $(-\pi, 0)$ and $(0, \pi)$, and satisfies

$$\lim_{\theta \to 0 \pm 0} f(\theta) = \mp\infty. \tag{41}$$

With no loss of generality we can take $f(-\pi + 0) = f(\pi - 0) = 0$. Let

$$\vartheta(x) := f^{-1}(-x), \quad x \in \mathbb{R} \setminus \{0\}. \tag{42}$$

Then ϑ is monotonically decreasing on $(-\infty, 0)$ and $(0, \infty)$, and

$$\vartheta(\pm\infty) = 0, \quad \vartheta(0 \pm 0) = \pm\pi.$$

Further, let

$$\Delta(n) = \begin{cases} \vartheta(n) - \vartheta(n+1), n = +0, 1, 2, \ldots, \\ \vartheta(n-1) - \vartheta(n), n = -0, -1, -2, \ldots \end{cases}$$

Consider the sequence of functions

$$\psi_n(s) = \frac{\vartheta(n) - \vartheta(n+s)}{\Delta(n)}, \quad s \in I := [-1/2, 1/2].$$

We assume that this sequence converges monotonically on I and that

$$\begin{aligned} \lim_{n \to +\infty} \psi_n(s) &= \psi(s), \\ \lim_{n \to -\infty} \psi_n(s) &= -\psi(-s), \end{aligned} \tag{43}$$

where the function ψ is monotonically increasing and continuous on I. Finally, we put

$$\xi(n) = \begin{cases} \vartheta(n+1)/\vartheta(n), n = +0, 1, 2, \ldots, \\ \vartheta(n-1)/\vartheta(n), n = -0, -1, -2, \ldots, \end{cases}$$

$$\alpha(n) = 1 - \xi(n).$$

We will need the following technical result.

Theorem 26. ([13, Proposition 5.6]) *Suppose the function ϑ has the form* (42) *and satisfies condition* (43). *Then*

$$\lim_{n \to \pm\infty} \frac{\Delta(n \pm 1)}{\Delta(n)} = d, \qquad 0 \le d \le 1. \tag{44}$$

Moreover,

$$\lim_{n \to \infty} \xi(n) = d, \quad \lim_{n \to \infty} \alpha(n) = 1 - d, \tag{45}$$

$$\lim_{n \to \infty} \frac{\alpha(n+1)}{\alpha(n)} = 1,$$

and

$$\psi(1/2) - d\psi(-1/2) = 1.$$

We will need the following two auxiliary functions

$$A(\theta) = \sum_{\vartheta(j)>\theta} \arctan(\alpha(j)) - \sum_{\vartheta(j)<-\theta} \arctan(\alpha(j)), \quad \theta \in (0,\pi), \tag{46}$$

and

$$C(n) = \sum_{j=m(n)}^{n-\sigma} \arctan \frac{\Delta(j)}{\vartheta(j)-\vartheta(n)} - \sum_{j=n+\sigma}^{M(n)} \arctan \frac{\Delta(j)}{\vartheta(n)-\vartheta(j)}, \tag{47}$$

where $\sigma = \operatorname{sign} n$, the number $m = m(n)$ is the j of the smallest modulus for which $|\vartheta(j)| \le \frac{3}{2}|\vartheta(n)|$, while $M = M(n)$ is the j of the largest modulus for which $|\vartheta(j)| \ge \frac{1}{2}|\vartheta(n)|$.

The quantity $A(\theta)$ relates the behaviour of $\vartheta(x)$ as $x \to +\infty$ to its behaviour as $x \to -\infty$; in other words, it connects the behaviour of f in a right semi-neighbourhood of zero to its behaviour in a left semi-neighbourhood (see (42)). The quantity $C(n)$ characterises the behaviour of $\vartheta(x)$ near the point $x = n$.

Theorem 27. ([13, Theorem 5.10]) *Suppose the function $a \in GL^\infty(\mathbb{T})$ is continuous on $\mathbb{T} \setminus \{1\}$ and has the form* (36) *with a function f that is monotonically increasing on $(-\pi,0)$ and $(0,\pi)$ and satisfies condition* (41). *In addition, assume that condition* (43) *is satisfied, that $d = 1$ in* (44), *and that the limits*

$$\lim_{\theta\to 0\pm 0} A(\theta) = a, \quad a \in \mathbb{R}, \tag{48}$$

$$\lim_{n\to\pm\infty} C(n) = 0 \tag{49}$$

exist, where $A(\theta)$ and $C(n)$ are defined by (46) *and* (47).

Then a admits the representation

$$a(t) = B(t)g(B(t))d(t), \tag{50}$$

which is a (B,p)-factorisation, with $g, d \in C(\mathbb{T})$. Moreover, the winding number of the function g is equal to zero, the Blaschke product B is constructed from the zeroes $z_j = r_j \exp\{i\vartheta(j)\}$, where $r_j = (1-\Delta(j)/2)/(1+\Delta(j)/2)$, $j = \pm1, \pm2, \ldots$, and the product

$$b(t) := g(B(t))d(t)$$

admits a p-factorisation of the form (5) *for any $1 < p < \infty$.*

Theorem 28. ([13, Theorem 5.12]) *Suppose the function f satisfies all the conditions of Theorem* 27 *and that in condition* (43)

$$\psi(s) = s. \tag{51}$$

Then the function a given by (36) *belongs to $H^\infty(\mathbb{T}) + C(\mathbb{T})$ and admits the representation*

$$a(t) = U(t)c(t), \tag{52}$$

where c is a continuous function on $\mathbb{T}$ and the inner function U has the form

$$U(t) = \frac{r_0 + B(t)}{1 + r_0 B(t)}, \quad r_0 = e^{-2},$$

with the same Blaschke product B as in (50).

Conditions (43), (48), (49), under which Theorems 27 and 28 hold, cover a very large class of symbols with arguments that increase in a neighbourhood of the discontinuity. However, they are not always easy to verify. The following theorems provide more convenient sufficient conditions. We assume as above that f is monotonically increasing on $[-\pi, 0)$ and $(0, \pi]$, and satisfies (41).

Theorem 29. ([13, Proposition 5.8]) *Let f be twice continuously differentiable on $[-\pi, \pi] \setminus \{0\}$ and let f' be monotonically decreasing* (*increasing*) *on* $(0, \pi)$ (*on* $(-\pi, 0)$ *respectively*) *and satisfy*

$$\lim_{\theta \to 0} \frac{f''(\theta)}{(f'(\theta))^2} = 0. \tag{53}$$

Then (43) *holds with the function $\psi(s) \equiv s$.*

It is not difficult to see that (53) implies

$$\lim_{\theta \to 0} \frac{1}{\theta f'(\theta)} = 0. \tag{54}$$

Theorem 30. ([13, Proposition 5.9]) *Suppose that f is twice continuously differentiable on $[-\pi, \pi] \setminus \{0\}$ and that f' is monotonically nonincreasing (nondecreasing) on* $(0, \pi)$ (*on* $(-\pi, 0)$ *respectively*) *and satisfies* (54) *and*

$$\lim_{\theta \to 0} \frac{f''(\theta)|\theta|^{1/2}}{(f'(\theta))^{3/2}} = 0. \tag{55}$$

Then (49) *holds.*

Theorem 31. ([13, Proposition 5.10]) *Suppose that all the assumptions of Theorem* 29 *are satisfied and that*

$$\left| \frac{f'(\theta)}{f'\left(\frac{3}{2}\theta\right)} \right| \le g, \tag{56}$$

where $g > 1$ does not depend on $\theta \in (-\pi, \pi) \setminus \{0\}$. Then (49) *holds.*

Theorem 32. ([13, Proposition 5.11]) *Suppose the function f is odd: $f(-\theta) = -f(\theta)$. Then* (48) *holds.*

Theorem 33. ([13, Proposition 5.12]) *Suppose f is continuously differentiable on $(-\pi,\pi)\setminus\{0\}$ and the function $\psi(\theta) = (\theta f'(\theta))^{-1}$ tends monotonically to zero as $\theta \to 0$. Then* (48) *holds whenever one of the following three conditions is satisfied:*

$$\int_0^\pi \left[f'(s) \arctan \frac{1}{sf'(s)} - f'(-s) \arctan \frac{1}{sf'(-s)} \right] ds < \infty; \tag{57}$$

$$\int_0^\pi \left| \frac{1}{(sf'(s))^2} - \frac{1}{(sf'(-s))^2} \right| \frac{ds}{s} < \infty; \tag{58}$$

$$\int_0^\pi \frac{1}{(sf'(s))^2} \frac{ds}{s} < \infty \quad \textit{and} \quad \int_0^\pi \frac{1}{(sf'(-s))^2} \frac{ds}{s} < \infty. \tag{59}$$

Below are several examples where the conditions of Theorem 27 are satisfied (see [13, Section 5.6]).

Example 1. Power whirls.

Consider the function

$$f(\theta) = \begin{cases} -c_+\theta^{-\lambda_+},\ \theta > 0, \\ c_-|\theta|^{-\lambda_-},\ \theta < 0, \end{cases}$$

where $c_\pm > 0$ and $\lambda_\pm \in (0,\infty)$. It obviously satisfies the conditions of Theorems 29 and 30 (and of Theorem 31), as well as condition (59), and consequently all the conclusions of Theorem 27 are valid for f.

One can consider a more general case that often arises in the theory of the Riemann-Hilbert problem with an infinite index

$$f(\theta) = \begin{cases} -c_+(\theta)\theta^{-\lambda_+},\ \theta > 0, \\ c_-(\theta)|\theta|^{-\lambda_-},\ \theta < 0, \end{cases} \tag{60}$$

where $\lambda_\pm > 0$, and the functions $c_\pm$ are continuous on $[0,\pi]$ and $[-\pi,0]$ respectively. Let us assume that $c_\pm(\theta)$ are twice continuously differentiable on $[-\pi,0)$ and $(0,\pi]$ and that

$$\lim_{\theta\to 0\pm 0} c'_\pm(\theta)\theta = 0, \qquad \lim_{\theta\to 0\pm 0} c''_\pm(\theta)\theta^2 = 0. \tag{61}$$

The conditions of Theorem 27 can be verified with the help of Theorems 29, 30, and 33.

Example 2. Power-logarithmic whirls. Now let

$$f(\theta) = \begin{cases} -c_+\theta^{-\lambda_+}(\log|\theta|^{-1})^{\beta_+},\ \theta > 0, \\ c_-|\theta|^{-\lambda_-}(\log|\theta|^{-1})^{\beta_-},\ \theta < 0, \end{cases}$$

where $c_\pm > 0$ and $\lambda_\pm \in (0, \infty)$, $\beta_\pm \in \mathbb{R}$. The applicability of Theorem 27 in this case is verified in the same way as in Example 1.

Example 3. Exponential and superexponential growth of the argument. Let

$$f(\theta) = \begin{cases} -c_+ \exp\{d_+ \theta^{-\lambda_+}\}, \ \theta > 0, \\ c_- \exp\{d_- |\theta|^{-\lambda_-}\}, \ \theta < 0, \end{cases}$$

where $c_\pm > 0$, $d_\pm > 0$ and $\lambda_\pm \in (0, \infty)$, or let

$$f(\theta) = \begin{cases} -c_+ \exp\{g_+ \exp(d_+ \theta^{-\lambda_+})\}, \ \theta > 0, \\ c_- \exp\{g_- \exp(d_- |\theta|^{-\lambda_-})\}, \ \theta < 0, \end{cases} \tag{62}$$

where $c_\pm > 0$, $d_\pm > 0$, $g_\pm > 0$, and $\lambda_\pm \in (0, \infty)$. The conditions of Theorem 27 are verified as in the preceding cases. Let us mention only that Theorem 31 does not apply here, while Theorem 30 does.

These examples show that the conditions of Theorem 27 are well suited to rapidly growing arguments $f(\theta)$. In particular, it is easy to see that a function f constructed via a composition of a finite number of exponentials similarly to (62) also satisfies (53), (55) and (59).

Let us now consider the case of slowly growing arguments $f(\theta)$.

Example 4. Logarithmic whirls.

Let

$$f(\theta) = \begin{cases} -c(\log \theta^{-1})^\beta, \ \theta > 0, \\ c(\log |\theta|^{-1})^\beta, \ \theta < 0, \end{cases} \tag{63}$$

where $\beta > 0$, $c > 0$. If $\beta > 1$, then f satisfies the conditions of Theorem 27, which can be verified by evaluating the limits (53), (55) and applying Theorem 32. On the other hand, if $\beta \in (0, 1]$, then f fails to satisfy the condition $d = 1$ in Theorem 27. The critical case $\beta = 1$ is the most important for us and we will consider it below (see Theorems 34, 35).

Similarly to Example 1, one can replace the constants $c_\pm$ in Examples 2–4 with continuous functions.

Example 5. Asymmetric whirls.

In Examples 1–3, condition (49) can be verified separately for left and right semi-neighbourhoods of the point $\theta = 0$ with the help of (59). This allows one to construct new examples that satisfy the conditions of Theorem 27 from the ones mentioned above by combining different types of whirling to the left and to the right. For instance, one can take

$$f(\theta) = \begin{cases} -c_+ \exp(g \exp(\theta^{-\lambda})), \ \theta > 0, \\ \quad c_- \log^\beta(|\theta|^{-1}), \qquad \theta < 0, \end{cases} \tag{64}$$

where $c_\pm > 0$, $g > 0$, $\lambda > 0$, and $\beta > 3/2$. The corresponding function (36) combines very fast oscillations to the right of the point $\theta = 0$ with very slow oscillations to the left of it.

It was mentioned in Example 4 that Theorem 27 does not cover the case of slow oscillations and that the natural boundary of its domain of applicability is the case of pure logarithmic whirls. In this case, we have the following result which was a key ingredient in the proof of Theorem 23.

Theorem 34. (See [13, Theorem 2.10 and the end of the proof of Theorem 5.9]) *Suppose $a \in GL^\infty(\mathbb{T})$ is continuous on $\mathbb{T} \setminus \{1\}$ and has the form* (36) *with a function f satisfying the condition*

$$\lim_{\theta\to\pm 0}\left(f(\theta) \pm \frac{1}{2}\log|\theta|^{-1}\right) = 0.$$

Then a admits the representation

$$a(t) = B(t)g(B(t))d(t), \tag{65}$$

where $g, d \in C(\mathbb{T})$, the winding number of g is 0, and B is the infinite Blaschke product with the zeroes

$$z_k = \frac{2 - \exp(-k + 1/2)}{2 + \exp(-k + 1/2)}.$$

Note by the way that B in the above theorem is an interpolating Blaschke product (see the end of Section 5.4 in [13]).

The following result is a generalisation of Theorem 34 (see [13, Section 5.7]).

Theorem 35. ([13, Theorem 5.11]) *Let a function $a \in GL^\infty(\mathbb{T})$ be continuous on $\mathbb{T}\setminus\{1\}$ and have the form* (36) *with a function f that is monotonically increasing on $(-\pi, 0) \cup (0, \pi)$ and satisfies the condition* (41). *Assume, in addition, that condition* (43) *is satisfied, that* (45) *holds with some number $0 < d < 1$, and that*

$$\lim_{n\to+\infty}\left(-\frac{\vartheta(n)}{\vartheta(-n)}\right) = 1.$$

Then the function a admits the following representation, which is a (B, p)-factorisation simultaneously for all $1 < p < \infty$:

$$a(t) = B(t)g(B(t))d(t),$$

where $g, d \in C(\mathbb{T})$. Moreover, the winding number of the function g is equal to zero and the Blaschke product B is constructed from the zeroes $z_j = r_j \exp\{i\vartheta(j)\}$, where

$$r_j = (1 - \Delta(j)/2)/(1 + \Delta(j)/2), \quad j = \pm 1, \pm 2, \ldots$$

Using a linear fractional transformation, one can easily transplant the above results from $\mathbb{T}$ to $\mathbb{R}$. The following analogue of a special case of Theorem 28 is of a direct relevance to Theorem 19.

Theorem 36. ([3, 4]) *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be an orientation preserving homeomorphism that is twice continuously differentiable for all sufficiently large values of $x > 0$ and is such that*

$$\liminf_{x\to+\infty} \frac{x\alpha''(x)}{\alpha'(x)} > -2, \tag{66}$$

$$\lim_{x\to+\infty} \frac{\alpha''(x)}{(\alpha'(x))^2} = 0, \tag{67}$$

$$\lim_{x\to+\infty} x^{1/2}\frac{\alpha''(x)}{(\alpha'(x))^{3/2}} = 0, \tag{68}$$

$$\lim_{x\to+\infty} (\alpha(x) + \alpha(-x)) = 0. \tag{69}$$

Then

$$e^{i\lambda\alpha} \in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}}), \quad \forall\lambda > 0.$$

Moreover the following representation holds

$$e^{i\lambda\alpha(x)} = B_\lambda(x)\, C_\lambda(x), \tag{70}$$

where B_λ is a Blaschke product with an infinite number of zeroes accumulating at infinity and C_λ is a unimodular function belonging to $C(\dot{\mathbb{R}})$.

Condition (66) is equivalent to the requirement that $x^2\alpha'(x)$ is strictly increasing for large values of x. Conditions (66)–(68) are satisfied for large classes of functions. Here are some examples:

$$\begin{aligned}
&\alpha(x) = c\, x^\gamma, \quad \gamma > 0,\\
&\alpha(x) = c \ln^\delta(x+1), \quad \delta > 1,\\
&\alpha(x) = c\, x^\gamma \ln^\delta(x+1), \quad \gamma > 0, \quad \delta \in (-\infty, \infty),\\
&\alpha(x) = c_1 \exp(c_2 x^\gamma), \quad \gamma > 0
\end{aligned}$$

with some positive constants c, c_1, c_2 (cf. Examples 1–4).

On the other hand, there are plenty of orientation preserving homeomorphisms $\alpha : \mathbb{R} \to \mathbb{R}$ for which $e^{i\alpha} \notin H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$. This is a consequence of the following result.

Theorem 37. ([1], [3]) *Given any orientation preserving homeomorphism $\eta : \mathbb{R} \to \mathbb{R}$, there exists an orientation preserving homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha - \eta \in L^\infty(\mathbb{R})$ and $e^{i\alpha} \notin H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$.*

Theorem 36 implies the following sufficient condition for (35) to hold.

Theorem 38. ([3]) *Suppose there exists $\delta > 1$ such that $\beta(x) := \alpha(x) - (\log x)^\delta$ is strictly increasing and twice continuously differentiable for all sufficiently large values of $x > 0$, and suppose β satisfies (66)–(68) (with β in* *place of α). Then $we^{i\lambda\alpha} \in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$ for all $\lambda > 0$, where w is the same as in* (10)–(11).

The final topic of this section is motivated by applications to the KdV equation (see Section 6). We are interested in conditions under which the argument of the quotient of two Blaschke products with purely imaginary zeroes in the upper half-plane is continuous on the real line. Consider the Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{z - ix_k}{z + ix_k}, \quad z \in \mathbb{C}_+ := \{z \in \mathbb{C} | \operatorname{Im} z > 0\} \tag{71}$$

with purely imaginary zeroes such that

$$x_1 > \cdots > x_k > x_{k+1} > \cdots > 0 \quad \text{and} \quad \lim_{k\to\infty} x_k = 0. \tag{72}$$

In this case, the standard Blaschke condition (see, e.g., [14, Ch. II, (2.3)] or [27, (13.13)]) reads

$$\sum_{k=1}^{\infty} x_k < \infty. \tag{73}$$

Theorem 24 takes the following simple form.

Theorem 39. *Let* $\arg B$ *denote the branch of the argument of the Blaschke product* (71)-(73) *which is continuous on* $\mathbb{R}\backslash\{0\}$ *and satisfies* $\lim_{x\to\pm\infty} \arg B(x) = 0$, *and let the branch of* arctan *be chosen so that* $\arctan x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. *Then* $\arg B$ *is increasing on* $\mathbb{R} \setminus \{0\}$,

$$\arg B(x) = -\arg B(-x), \quad x \in \mathbb{R}, \tag{74}$$

$$\lim_{x\to\pm 0} \arg B(x) = \mp\infty, \tag{75}$$

and

$$\arg B(x) = -2\sum_{k=1}^{\infty} \arctan \frac{x_k}{x}, \quad x \neq 0. \tag{76}$$

Let f_B be a continuous and decreasing on $(0, +\infty)$ function such that

$$f_B(k) = x_k.$$

Let $\Delta_k = x_k - x_{k+1}$ and $\Delta_k^{(2)}(s) = f_B(k+s) - f_B(k) + s(x_k - x_{k+1})$, $s \in [-1/2, 1/2]$.

Theorem 40. ([20]) *Suppose the sequence* $\{x_k\}$ *is such that*

$$\lim_{k\to\infty} \frac{x_k - x_{k+1}}{x_k} = 0 \tag{77}$$

and

$$\lim_{k\to\infty} \sup_{s\in[-1/2,1/2]} \left(\frac{|\Delta_k^{(2)}(s)|}{\Delta_k} \right) = 0. \tag{78}$$

Then

$$\arg B(x) = -2 \int_{1/2}^{\infty} \arctan \frac{f_B(u)}{x} du + O(x), \tag{79}$$

where $\lim_{x\to 0} O(x) = 0$.

Theorem 41. ([20]) *Let a function* f_B *be continuously differentiable on* $(0, +\infty)$ *and satisfy all the conditions of Theorem 40. Then, for* $x > 0$

$$\arg B(x) = -2x \int_0^1 \frac{\varphi_B(y)}{x^2 + y^2} dy + \frac{\pi}{2} + O_1(x), \tag{80}$$

where $\varphi_B(y) := f_B^{-1}(y)$ *is the inverse function of* f_B *and*

$$\lim_{x\to 0} O_1(x) = 0.$$

Let now $R(x) = B_1(x)/B_2(x)$, where $B_1(x)$ and $B_2(x)$ are Blaschke products with the zeroes $if_{B_j}(k)$, $j = 1, 2$, where the functions f_{B_j} satisfy the conditions of Theorem 41. Introduce the function

$$r(y) := \varphi_{B_1}(y) - \varphi_{B_2}(y),$$

where $\varphi_{B_j}(y) := f_{B_j}^{-1}(y)$, $j = 1, 2$.

Theorem 42. ([20]) *Suppose at least one of following two conditions holds:*

i)

$$r(y) = r_0 + O_2(y)$$

with some $r_0 \in \mathbb{R}$, $\quad \lim_{y\to 0} O_2(y) = 0$;

ii)

$$\int_0^y r(s) ds = r_1 y + O_3(y)$$

with some $r_1 \in \mathbb{R}$, $\quad \lim_{v\to 0} \left(\frac{O_3(y)}{y} \right) = 0.$

Then

$$\arg R(x) = r_2 + O_4(x),$$

with some $r_2 \in \mathbb{R}$, $\quad \lim_{x\to 0} O_4(x) = 0.$

The following corollary of Theorem 42 together with Theorem 36 play an important rôle in the proof of Theorem 43.

Corollary. ([20]) *Let*

$$B_j(z) = \prod_{k=1}^{\infty} \frac{z - ix_k^{(j)}}{z + ix_k^{(j)}}, \quad j = 1, 2$$

be two Blaschke products with interlacing $\left(x_k^{(1)} > x_k^{(2)} > x_{k+1}^{(1)}\right)$ *imaginary zeroes accumulating at* 0*, and let* f *be a real continuously differentiable function such that* $f(2x)$ *and* $f(2x-1)$ *satisfy the conditions of Theorem 41 and*

$$f(k) = \begin{cases} x^{(1)}_{\frac{k+1}{2}}, & k \text{ is odd,} \\ \\ x^{(2)}_{\frac{k}{2}}, & k \text{ is even.} \end{cases}$$

Then $\arg B_1/B_2$ *is continuous on the real line.*

6 Applications to the KdV Equation

Let P be the projection defined by (2), $Q := I - P$ and let

$$(Jf)(x) = f(-x) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \tag{81}$$

be the reflection operator. The Hankel operator with the symbol $a \in L^\infty(\mathbb{R})$ is defined by the formula

$$(\mathbb{H}(a)f)(x) := (JQaf)(x) : H^2(\mathbb{R}) \to H^2(\mathbb{R}). \tag{82}$$

The symbol

$$\phi(x) = e^{i\left(tx^3 + cx\right)} d(x), \quad t > 0, \ c \in \mathbb{R} \tag{83}$$

arises in the inverse scattering transform method for the Korteweg-de Vries (KdV) equation (see [29], [30]). The form of the unimodular function $d(x)$ depends on the properties of the initial data for the KdV equation. In certain important cases the function d has the form

$$d(x) = \frac{B_1(x)}{B_2(x)} I(x), \tag{84}$$

where B_1, B_2 are Blaschke products with zeroes converging to 0 along the imaginary axis and I is an inner function ($I \in H^\infty(\mathbb{R})$ and $|I(x)| = 1$ a.e. on the real line).

The proof of the following result relies on Theorems 7, 36 and 42.

Theorem 43. ([20]) *Let* $\phi(x) = e^{i\left(tx^3+cx\right)}\frac{B_1(x)}{B_2(x)}I(x)$, $t > 0$, $c \in \mathbb{R}$, *where* B_j, $j = 1, 2$ *are Blaschke products with zeroes* $\{if_{B_j}(k)\}$ *and the real-valued functions* f_{B_j}, $j = 1, 2$ *satisfy the conditions of Theorems 40–42. Then the Toeplitz operator*

$$T(\phi) : H^2(\mathbb{R}) \to H^2(\mathbb{R})$$

is left invertible, the Hankel operator

$$\mathbb{H}(\phi) : H^2(\mathbb{R}) \to H^2(\mathbb{R})$$

is compact and the operator

$$I + \mathbb{H}(\phi) : H^2(\mathbb{R}) \to H^2(\mathbb{R})$$

is invertible.

Theorem 43 plays a crucial rôle in the proof of case 3 in the following theorem. Consider the Cauchy problem for the Korteweg-de Vries equation

$$\frac{\partial u\,(x,t)}{\partial t} - 6u\,(x,t)\,\frac{\partial u\,(x,t)}{\partial x} + \frac{\partial^3 u\,(x,t)}{\partial x^3} = 0, \quad t \geq 0,\ x \in \mathbb{R}, \tag{85}$$

$$u\,(x,0) = q(x), \tag{86}$$

and the Schrödinger operator $H_q = -d^2/dx^2 + q\,(x)$ on $L^2\,(\mathbb{R})$. Let $H_q^D = -d^2/dx^2 + q\,(x)$ be the corresponding operator on $L^2\,(-\infty, 0)$ with the Dirichlet boundary condition $u\,(0) = 0$.

Theorem 44. ([20]) *Assume that the initial profile* $q\,(x)$ *in (86) is real-valued, locally integrable, supported in* $(-\infty, 0)$ *and such that*

$$\inf \operatorname{Spec}\,(H_q) = -a^2 > -\infty. \tag{87}$$

Then the Cauchy problem for the KdV equation (85)-(86) has a unique solution $u\,(x,t)$ *which is a meromorphic function in* x *on the whole complex plane with no real poles for any* $t > 0$ *if at least one of the following conditions holds:*

1. *The operator* H_q^D *has a non-empty absolutely continuous spectrum;*
2. $\operatorname{Spec}\left(H_q^D\right) \cap \mathbb{R}_-$ *is a set of uniqueness of an* $H^\infty(\mathbb{R})$ *function;*
3. $\left\{\operatorname{Spec}\left(H_q^D\right) \cup \operatorname{Spec}\,(H_q)\right\} \cap \mathbb{R}_-$ *is a discrete set* $\{-x_n^2\}_{n\geq 1}$ *such that the sequence* $\{x_n\}_{n\geq 1}$ *satisfies the conditions of the Corollary of Theorem 42.*

7 Some Open Problems

There are of course many open problems in the spectral theory of Toeplitz operators. Here we list just a few of them, mainly those that are directly related to the topics discussed above.

1. Describe inner functions/Blaschke products v for which $\rho \in A_p \implies \rho \circ v \in A_p$ (cf. Theorems 14 and 15). In particular, is the condition (27) necessary for this implication to hold in the case of Blaschke products with positive zeroes? Perhaps one should try to describe pairs (ρ, v), where $\rho \in A_p$ and v is an inner function, such that $\rho \circ v \in A_p$.
2. Find conditions on an orientation preserving homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ that are necessary and sufficient for

$$e^{i\lambda\alpha} \in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}}), \quad \forall \lambda > 0$$

   to hold (cf. Theorem 36).
3. According to Theorem 16, $a \in \mathrm{fact}(\infty, p)$ is a sufficient condition for the the right/left invertibility of $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$. Is this also a necessary condition for the right/left invertibility or even for the $\Phi_\pm$ property of $T(a)$? The answer is positive for $p = 2$ (see [13, Section 2.7]).
4. Study spectral properties of $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ when a belongs to a Douglas algebra, i.e. to a closed subalgebra of $L^\infty(\mathbb{T})$ containing $H^\infty(\mathbb{T})$ (cf. Theorem 7) . According to the Chang–Marshall theorem, every such algebra is generated by $H^\infty(\mathbb{T})$ and the complex conjugates of some inner functions (see, e.g., [14, Ch. IX]).

Finally, we would like to reiterate that very little is known about the (essential) spectrum of $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ for a general $a \in L^\infty(\mathbb{T})$.

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