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EIGENVALUES OF HESSENBERG TOEPLITZ MATRICES GENERATED BY SYMBOLS WITH SEVERAL SINGULARITIES *

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Abstract. In a recent paper, we established asymptotic formulas for the eigenvalues of the $n\times n$ truncations of certain infinite Hessenberg Toeplitz matrices as n goes to infinity. The symbol of the Toeplitz matrices was of the form $a(t)=t^{-1}(1-t)^{\alpha}f(t)$ $(t\in\mathbb{T})$, where α is a positive real number but not an integer and f is a smooth function in H^{∞}. Thus, a has a single power singularity at the point 1. In the present work we extend the results to symbols with a finite number of power singularities. To be more precise, we consider symbols of the form $a(t)=t^{-1}f(t)\prod_{k=1}^{K}(1-t/t_k)^{\alpha_k}$ $(t\in\mathbb{T})$, where $t_k=e^{i\theta_k}$, the arguments θ_k are all different, and the exponents α_k are positive real numbers but not integers.
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1 Introduction and main results

Given a function $a\in L^1$ on the unit circle in the complex plane $\mathbb{T}$, we denote by

$$a_k=\int_0^{2\pi}a\left(e^{i\theta}\right)e^{-ik\theta}\,d\theta/2\pi,\quad k\in\mathbb{Z},$$

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the kth Fourier coefficient and by $T_n(a)$ the $n \times n$ Toeplitz matrix $(a_{j-k})_{j,k=1}^n$. We are interested in the behavior of the eigenvalues of $T_n(a)$ as n goes to infinity. The function a is usually referred to as the symbol or the generating function of the sequence $\{T_n(a)\}_{n=1}^{\infty}$.

For real-valued functions a the matrices $T_n(a)$ are all Hermitian and a number of results on the asymptotics of the eigenvalues of $T_n(a)$ are available in this case: see, for example, [6], [12], [15], [17], [19], [20], [21], [22], [24], [25], [27], [28]. In this case the eigenvalues mimic in the one or other sense the distribution of the values of the function a at equispaced points on the unit circle.

The picture is less complete for complex-valued symbols. Papers [10], [14], [18] are devoted to the limiting behavior of the eigenvalues of $T_n(a)$ if a is a rational function, while papers [1] and [26] embark on the asymptotic eigenvalue distribution in the case of non-smooth symbols. In [23] and [26], it was observed that if $a \in L^\infty$ and the essential range $\mathcal{R}(a)$ does not separate the plane, then the eigenvalues of $T_n(a)$ approximate $\mathcal{R}(a)$, which resembles the Hermitian case. Many of the results of the papers cited above can also be found in the books [5], [7], [8].

An extreme situation is the one where $a_k = 0$ for $k \le -1$. Then, the matrices $T_n(a)$ are lower triangular and hence the spectrum $\operatorname{sp} T_n(a)$ is just the singleton $\{a_0\}$. Note that a_0 captures almost no information about the function a itself. The first interesting case beyond this trivial situation is the one where $T_n(a)$ has an additional super-diagonal and thus is a Hessenberg Toeplitz matrix. Of course, this happens if and only if $a_k = 0$ for $k \le -2$. Such symbols can be analytically continued into the punctured disk $0 < |z| < 1$, which, as pointed out in [18] and [26], can result in an eigenvalues distribution along points and curves that are very different from the range $\mathcal{R}(a)$. On the other hand, the presence of singularities in the symbol causes the opposite tendency, that is, it somehow forces the eigenvalues to mimic the range [26].

In [4], we considered symbols with a singularity of the type $(1-t)^\alpha$ $(t \in \mathbb{T})$ in order to illustrate certain instability phenomena in the eigenvalue distribution. The eigenvalues of the Hessenberg Toeplitz matrices generated by $a(t) = t^{-1}(1-t)^\alpha$ were studied in [2]. The recent papers [9] and [16] contain intriguing numerical experiments for individual eigenvalues of Toeplitz matrices whose symbols have a so-called Fisher-Hartwig singularity. These are special symbols that are smooth on $\mathbb{T}$ minus a single point but not smooth on the entire circle $\mathbb{T}$; see [7], [8]. Papers [9] and [16] motivated us to take up the singularity $(1-t)^\alpha$ again, and in [3] we established fairly precise results on the eigenvalues of $T_n(a)$ in the case where $a(t) = t^{-1}(1-t)^\alpha f(t)$ and f satisfies certain smoothness and analyticity requirements. In the present paper, we generalize these results to symbols with several singularities of the power type.

Let H^∞ be the usual Hardy space of (boundary values of) bounded analytic functions in the unit disk $\mathbb{D}$. Given $a \in C(\mathbb{T})$, we denote by $\operatorname{wind}_\lambda(a)$ the winding number of a about a point $\lambda \in \mathbb{C} \setminus \mathcal{R}(a)$ and by $\mathcal{D}(a)$ the set of all $\lambda \in \mathbb{C} \setminus \mathcal{R}(a)$ for which $\operatorname{wind}_\lambda(a) \ne 0$. In this paper we study the eigenvalues of $T_n(a)$ for symbols $a(t) = t^{-1} f(t) \prod_{k=1}^{K} (1 - t/t_k)^{\alpha_k}$ $(t \in \mathbb{T})$, where f is a smooth function subject to additional conditions, the points $t_k = e^{i\theta_k}$ are all different, and the exponents α_k are distinct positive real numbers but not integers. Thus, we require in particular that $\alpha_k \ne \alpha_\ell$ for $k \ne \ell$. Our approach also works if two or more of the exponents α_k coincide, although then a series of technical details emerges. To keep this paper within a reasonable volume, we decided not to embark on the case of coinciding exponents here.

We enumerate the singularity points t_k as follows: let t_1 be such that $\alpha_1 = \min_{1 \le k \le K}\{\alpha_k\}$ and number the remaining t_k counterclockwise. Let $\{T_k\}_{k=1}^{K}$ be the connected components of $\mathbb{T} \setminus \{t_1, \ldots, t_K\}$ and denote by $\operatorname{clos} T_k$ be the arc T_k together with its two endpoints. Let $h(t) := a(t)t$ and h_0 be its zeroth Fourier coefficient. We assume that a has the following properties.

1. $h \in H^\infty$ and $h_0 \ne 0$.

2. $f \in C^\infty(\mathbb{T})$.

3. h can be analytically extended to an open neighborhood W of $\mathbb{T} \setminus \{t_1, \ldots, t_K\}$ not containing the set $\{t_1, \ldots, t_K\}$.

4. The derivative $a'(t)$ does not vanish for $t \in \mathbb{T} \setminus \{t_1, \ldots, t_K\}$, each $a(\operatorname{clos} T_k)$ is a Jordan curve which surrounds the points in its interior clockwise, and for $k \ne \ell$, the interiors of the curves $a(\operatorname{clos} T_k)$ and $a(\operatorname{clos} T_\ell)$ are disjoint.

Figure 2 shows two concrete examples of such functions.

If f is identically 1, that is, if $a(t) = t^{-1}\prod_{k=1}^{K}(1-t/t_k)^{\alpha_k}$, then properties 1 to 4 are satisfied if and only if $\sigma := \sum_{k=1}^{K}\alpha_k < 2$. To see this, let t revolve the unit circle once counterclockwise starting at t_1. We have

$$a(t) = t^{-1}(1-t/t_1)^{\sigma}\prod_{k=2}^{K}\left(\frac{1-t/t_k}{1-t/t_1}\right)^{\alpha_k}.$$

Taking into account that the argument of $(1-t/t_k)/(1-t/t_1)$ is piecewise constant and that $t^{-1}(1-t/t_1)^{\sigma}$ describes a loop that encircles the points in its interior exactly once clockwise if and only if $\sigma < 2$, it is not difficult to see that the range of a is a flower with K non-overlapping petals and that the petals surround their interiors exactly once clockwise if and only if $\sigma < 2$.

Let $D_n(a)$ denote the determinant of $T_n(a)$. Thus, the eigenvalues λ of $T_n(a)$ are the solutions of the equation $D_n(a-\lambda) = 0$. Our assumptions imply that $T_n(a)$ is a Hessenberg matrix, that is, it arises from a lower triangular matrix by adding the super-diagonal. This circumstance together with the Baxter-Schmidt formula for Toeplitz determinants allows us to express $D_n(a-\lambda)$ as a Fourier integral. The value of this integral mainly depends on λ and on the singularity of each $(1-t/t_k)^{\alpha_k}$ at the point t_k. Let W_0 be a small open neighborhood of zero in $\mathbb{C}$. We show that for every point $\lambda \in \mathcal{D}(a) \cap \big(a(W)\setminus W_0\big)$ there is a unique point $t_\lambda \notin \overline{\mathbb{D}}$ such that $a(t_\lambda) = \lambda$. After exploring the contributions of λ and the singular points t_k to the Fourier integral, we get the following asymptotic expansion for $D_n(a-\lambda)$.

Theorem 1.1. *Let $a(t) = t^{-1}h(t)$ be a symbol with properties 1 to 4. Then, for every small open neighborhood W_0 of zero in $\mathbb{C}$, every $\lambda \in \mathcal{D}(a)\cap\big(a(W)\setminus W_0\big)$, and every real positive μ,*

$$D_n(a-\lambda) = (-h_0)^{n+1}\left(\frac{1}{t_\lambda^{n+2}a'(t_\lambda)} - \sum_{(k,\ell,s)\in\mathcal{L}_\mu}\frac{A_{k,\ell,s}}{\lambda^{s+1}t_k^n n^{\alpha_k s+\ell+1}} + R_1(\lambda,n)\right), \tag{1.1}$$

where $\mathcal{L}_\mu$ is the collection of all the triples (k,ℓ,s) such that $k\in\{1,\dots,K\}$, $\ell\in\{0,1,\dots\}$, $s\in\{1,2,\dots\}$, and $\alpha_k s+\ell+1<\mu$;

$$A_{k,\ell,s} = \frac{\sin(\alpha_k\pi s)\Gamma(\alpha_k s+\ell+1)}{i^\ell \pi t_k^{s+1}\ell!}\left[\frac{f^s(t_k e^{i\theta})g^{\alpha_k s}(\theta)\prod_{j\neq k}(1-e^{i\theta}t_k/t_j)^{\alpha_j s}}{e^{i\theta(s+1)}}\right]_{\theta=0}^{(\ell)},$$

$g(\theta) = (e^{i\theta}-1)/(i\theta)$, and $R_1(\lambda,n) = O(1/n^\mu)$ as $n\to\infty$, uniformly with respect to $\lambda\in a(W)\setminus W_0$.

Of course, in Theorem 1.1 the superscript (ℓ) means "take ℓ derivatives with respect to θ" and the subscript $\theta = 0$ means "evaluate in $\theta = 0$".

The order of the sum in (1.1) is $1/n^{\alpha_1+1}$. Thus, among the singularities of the symbol a, the factor $(1-t/t_1)^{\alpha_1}$ makes the biggest contribution to $D_n(a-\lambda)$. Changing to the variable t/t_1 in a, we can obtain a new symbol $\tilde{a}$ in which the first singularity point will be 1. Moreover, $\operatorname{sp} T_n(a) = \operatorname{sp} T_n(\tilde{a})$; see [18] or [5, Section 11.1] for details. In order to simplify some of our forthcoming results, we henceforth assume without loss of generality that $t_1 = 1$.

Let $\omega_n := \exp(-2\pi i/n)$ and $\mathcal{J}_n := \big\{j\in\{0,\dots,n-1\} : a(\omega_n^j)\notin W_0\big\}$, also let $\gamma := \min_{1\le k\le K}\{\alpha_k : \alpha_k > \alpha_1\}$ and $\zeta := \min\{1,\alpha_1,\gamma-\alpha_1\}$. As μ is any real positive number, we can develop (1.1) with an arbitrary error bound, but to make our calculations reasonable and readable, we use Theorem 1.1 with $\mu = 2\zeta+\alpha_1+1$ to obtain the following two results.

Theorem 1.2. *Let $a(t) = t^{-1}h(t)$ be a symbol with properties 1 to 4. Then, for every small open neighborhood W_0 of the origin in $\mathbb{C}$ and every $j\in\mathcal{J}_n$, the equation $D_n(a-\lambda) = 0$ has a solution $\lambda = \lambda_{j,n}$ such that*

$$\begin{aligned} t_{\lambda_{j,n}} = \omega_n^j n^{(\alpha_1+1)/n}\Bigg(1 &+ \sum_{m=1}^{[1+2\zeta]}\log^m\!\left(\frac{a^2(\omega_n^j)}{A_{1,0,1}\omega_n^{2j}a'(\omega_n^j)}\right)\frac{1}{m!\,n^m} \\ &- \frac{1}{A_{1,0,1}}\sum_{(k,\ell,s)\in\mathcal{K}}\frac{A_{k,\ell,s}}{t_k^n a^{s-1}(\omega_n^j)n^{\alpha_k s+\ell-\alpha_1+1}} + R_2(j,n)\Bigg), \end{aligned} \tag{1.2}$$

where $\mathcal{K}$ is the collection of all triples $(k,\ell,s) \neq (1,0,1)$ such that $k \in \{1,\dots,K\}$, $\ell \in \{0,1,\dots\}$, $s \in \{1,2,\dots\}$, and $\alpha_k s + \ell < 2\zeta + \alpha_1$. The remainder satisfies

$$R_2(j,n) = O(1/n^{2\zeta+1}) + O(\log n/n^2)$$

as $n \to \infty$, uniformly in $j \in \mathcal{J}_n$.

The terms $\log^m(\cdot)/(m!\,n^m)$ are large when ω_n^j is close to one of the singularity points t_j and are small when ω_n^j is far from all the t_j's. Thus, these terms correct the behavior of the eigenvalues close to each singularity point.

Theorem 1.3. *Let $a(t) = t^{-1}h(t)$ be a symbol with properties 1 to 4. Then, for every small neighborhood W_0 of zero in $\mathbb{C}$ and every $j \in \mathcal{J}_n$,*

$$\begin{aligned}\lambda_{j,n} &= a(\omega_n^j) + (\alpha_1+1)\omega_n^j a'(\omega_n^j)\frac{\log n}{n} \\ &\quad + \omega_n^j a'(\omega_n^j) \sum_{m=1}^{[1+2\zeta]} \log^m\left(\frac{a^2(\omega_n^j)}{A_{1,0,1}\omega_n^{2j} a'(\omega_n^j)}\right)\frac{1}{m!\,n^m} \\ &\quad - \frac{\omega_n^j a'(\omega_n^j)}{A_{1,0,1}} \sum_{(k,\ell,s)\in\mathcal{K}} \frac{A_{k,\ell,s}}{t_k^n a^{s-1}(\omega_n^j) n^{\alpha_k s+\ell-\alpha_1+1}} + R_3(j,n),\end{aligned} \tag{1.3}$$

where ζ and $\mathcal{K}$ are as in Theorem 1.2 and

$$R_3(j,n) = O(1/n^{2\zeta+1}) + O(\log^2 n/n^2)$$

as $n \to \infty$, uniformly in $j \in \mathcal{J}_n$.

Figures 1 and 2 illustrate Theorem 1.3.

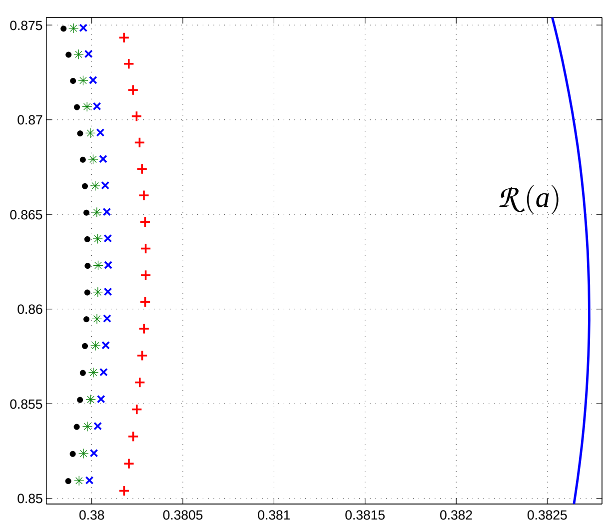

Figure 1. The picture shows a piece of $\mathcal{R}(a)$ for the symbol $a(t) = t^{-1}(1-t)^{0.3}(1-t/e^{2i})^{0.4}(1-t/e^{4i})^{0.5}$ (solid blue line) located far from zero. The black dots are $\operatorname{sp} T_{4096}(a)$ calculated by *Matlab*. The red pluses, blue crosses, and green stars are the approximations obtained by using 2, 3, and 4 terms of (1.3), respectively.

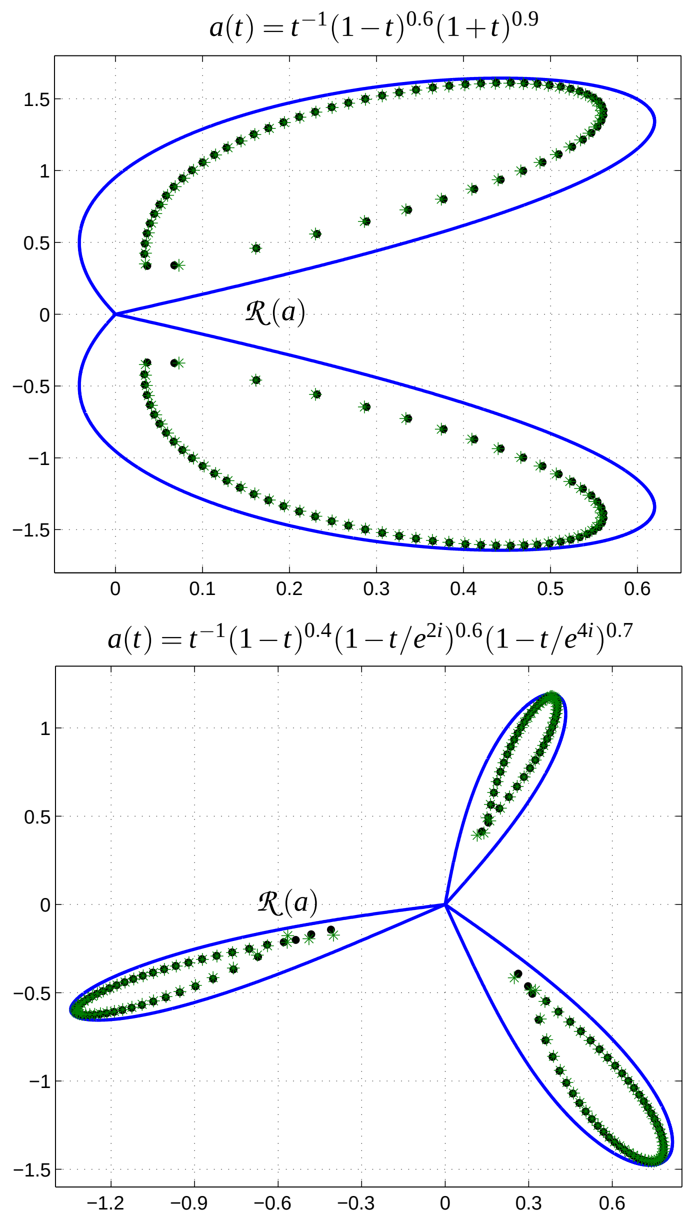

Figure 2. The black dots and the green stars, are the spectrum of $T_{128}(a)$ calculated with *Matlab* and formula (1.3) with 4 terms, respectively.

2 Toeplitz determinant

Consider the function $b^{(\lambda)}(t) := 1/\big(h(t) - \lambda t\big)$ where $\lambda \in \mathcal{D}(a)$ and $t \in \mathbb{T}$.

Lemma 2.1. *Let $a(t) = t^{-1}h(t)$ be a symbol with property 1. Then, for each $\lambda \in \mathcal{D}(a)$ and every $n \in \mathbb{N}$,*

$$D_n(a-\lambda) = (-1)^n h_0^{n+1} b_n^{(\lambda)}, \tag{2.1}$$

where $b_n^{(\lambda)}$ stands for the nth Fourier coefficient of $b^{(\lambda)}$ and h_0 for the zeroth Fourier coefficient of h.

Proof. The Baxter-Schmidt formula, which can for example be found in [5, p. 37], says that if $n, r \geq 1$ are integers and f is a function which is analytic and non-zero in some neighborhood of the origin, then

$$f_0^{-r} D_n(t^{-r} f) = (-1)^{rn} [1/f]_0^{-n} D_r(t^{-n}/f),$$

where $[\,]_n$ denotes the nth Fourier coefficient. Because of property 1, the function $f(t) := h(t) - \lambda t$ satisfies the hypothesis of the Baxter-Schmidt formula. Finally, taking $r = 1$ we easily obtain the lemma. □

With the aid of expression (2.1) we will calculate the Toeplitz determinant $D_n(a-\lambda)$ as a Fourier integral. As in the one singularity case [3], this is our starting point to find an asymptotic expansion for the eigenvalues of $T_n(a)$. The major contributions to this integral comes from λ when λ is close to $\mathcal{R}(a)$ and from the singularity points t_k. We analyze them in separate sections.

3 Contribution of λ to the asymptotic behavior of D_n

Recall that

$$b_n^{(\lambda)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} b^{(\lambda)}\big(e^{i\theta}\big) e^{-in\theta} d\theta,$$

is the nth Fourier coefficient of the function $b^{(\lambda)}$.

Lemma 3.1. *Let $a(t) = t^{-1}h(t)$ be a symbol satisfying properties 1, 3, and 4. Let W_0 be a small open neighborhood of zero in $\mathbb{C}$. Then, for each $\lambda \in \mathcal{D}(a) \setminus W_0$ sufficiently close to $\mathcal{R}(a)$, there is a unique point t_λ in $W \setminus \overline{\mathbb{D}}$ such that $a(t_\lambda) = \lambda$. Moreover, the point t_λ is a simple pole for $b^{(\lambda)}$.*

Proof. Enumerate the collection $\{T_k\}_{k=1}^{K}$ in the following way: for $1 \leq k < K$ let T_k be such that t_k and t_{k+1} are its extreme points, and let T_K be such that t_K and $t_1 = 1$ are its extreme points. The symbol a maps each arch T_k to a different petal $P_k := a(T_k)$ in $\mathcal{R}(a)$; see Figure 3. As h belongs to H^∞ and can be analytically extended to W, the map h can be thought of as a bounded and analytic function in $\mathbb{D} \cup W$. Since $h_0 = h(0) \neq 0$, the function $z^{-1}h(z) = a(z)$ is unbounded in $\mathbb{D}$. Thus, the map a must send $\mathbb{D} \setminus \{0\}$ to the exterior of $\mathcal{R}(a)$, that is, the unbounded connected component of $\mathbb{C} \setminus \mathcal{R}(a)$, and it must accordingly send $W \setminus \overline{\mathbb{D}}$ to $\mathcal{D}(a) \cap a(W)$.

By property 4, $a'(t) \neq 0$ for every $t \in T_k$. Take $S = \{t \in T_k : a(t) \notin W_0\}$. As a' is also analytic in W, for each $t \in S$ there is an open neighborhood $V_t^{(k)} \subset W$ of t such that $a'(t) \neq 0$ for every $t \in V_t^{(k)}$. Then, there is an open neighborhood $U_t^{(k)} \subset V_t^{(k)}$ of t such that a is a conformal map (and hence bijective) from $U_t^{(k)}$ to $a\big(U_t^{(k)}\big)$. As each S is compact, we can take a finite sub-cover from $\big\{U_t^{(k)}\big\}_{t \in S}$, say $U^{(k)} := \bigcup_{s=1}^{N_k} U_{t_s}^{(k)}$. It follows that a is a conformal map (and hence bijective) from $U^{(k)} \supset S^{(k)}$ to $a\big(U^{(k)}\big) \supset a\big(S^{(k)}\big)$.

Let $U := \bigcup_{k=1}^{K} U^{(k)}$. The lemma holds for every $\lambda \in a(U) \cap (\mathcal{D}(a) \setminus W_0)$. Finally, since $a'(t_\lambda) \neq 0$, the point t_λ must be a simple pole of $b^{(\lambda)}$. □

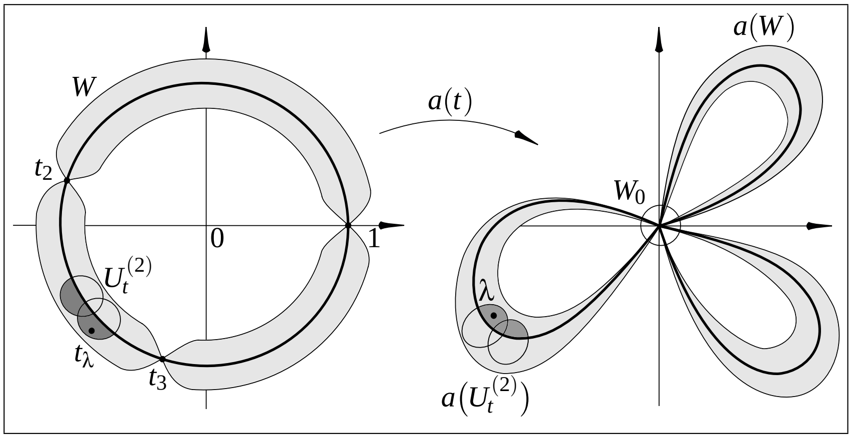

Figure 3. A typical range for the map a with 3 singularities over the unit circle.

Lemma 3.1 allows us to write

$$b^{(\lambda)}(z) = \frac{1}{t_\lambda a'(t_\lambda)(z-t_\lambda)} + \hat{b}^{(\lambda)}(z), \tag{3.1}$$

where $\hat{b}^{(\lambda)}$ is analytic with respect to z in W and uniformly bounded with respect to λ in $a(W)\setminus W_0$. Taking Fourier coefficients and writing $\hat{b}^{(\lambda)}(\theta)$ instead of $\hat{b}^{(\lambda)}\left(e^{i\theta}\right)$, we easily obtain

$$b_n^{(\lambda)} = \frac{-1}{t_\lambda^{n+2} a'(t_\lambda)} + I, \tag{3.2}$$

where

$$I := \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{b}^{(\lambda)}(\theta) e^{-in\theta} d\theta.$$

The first term in (3.2) times $(-1)^n h_0^{n+1}$ is the contribution of t_λ to the asymptotic expansion of $D_n(a-\lambda)$; see (2.1). The function $\hat{b}^{(\lambda)}$ has singularities at each θ_k, and we use this fact to expand I in the following Section.

4 Contribution of t_k to the asymptotic behavior of D_n

We start this Section by constructing a particular partition of the unity. Let δ be a small number satisfying $0 < \delta < \min_{j\neq k}\{|\theta_j - \theta_k|\}/2$ and take a function $\Phi_0 \in C^\infty[-\pi,\pi]$ which is supported in $(-\delta/2,\delta/2)$ and is identically 1 in $(-\delta/4,\delta/4)$. We may also suppose that $\mathcal{R}(\Phi_0) = [0,1]$.

For each $x \in [-\pi,\pi]$, let $\Phi_x(\theta) := \Phi_0(\theta - x)$. The collection

$$\mathcal{P} := \{\Phi_{\theta_1}, \ldots, \Phi_{\theta_K}, \Phi^*\},$$

with $\Phi^*(\theta) := 1 - \sum_{k=1}^{K} \Phi_{\theta_k}(\theta)$, is a partition of the unity for the interval $[-\pi,\pi]$. By pasting segments $[-\pi,\pi]$ in both directions, we continue this partition $\mathcal{P}$ to the entire real line $\mathbb{R}$.

We will use the following well known asymptotic results, which are, for example, in [11, p. 47] and [13, p. 97], respectively.

Theorem 4.1. *If $\alpha < \beta$, $v \in C^K[\alpha,\beta]$, and $v^{(s)}(\alpha) = v^{(s)}(\beta) = 0$ for $0 \le s \le K$, then*

$$\int_\alpha^\beta v(\theta) e^{-in\theta} d\theta = \frac{1}{(in)^K}\int_\alpha^\beta v^{(K)}(\theta) e^{-in\theta} d\theta = o(1/n^K) \quad \text{as } n \to \infty.$$

Theorem 4.2. *Let $\beta > 0$, $\delta > 0$, $v \in C^\infty[0,\delta]$, and $v^{(s)}(\delta) = 0$ for all $s \geq 0$. Then, for every $K \in \mathbb{N}$,*

$$\int_0^\delta \theta^{\beta-1} v(\theta) e^{in\theta} d\theta = \sum_{k=0}^{K-1} \frac{v^{(k)}(0)\Gamma(\beta+k) i^{\beta+k}}{k! n^{\beta+k}} + R_{K,v}(n),$$

where $|R_{K,v}(n)| \leq C_{K,v}/n^{\beta+K}$, the branch of the power $\beta+k$ is the one corresponding to the argument in $(-\pi,\pi]$, and $\Gamma(z)$ is Euler's Gamma function. If v depends on a parameter and the L^∞ norms of the derivatives $v^{(s)}$ for $0 \leq s \leq K$ have bounds that do not depend on the parameter, then one can take a single constant $C_{K,v}$ for all parameters.

Lemma 4.3. *For every sufficiently small positive δ, we have*

$$I = \frac{1}{2\pi} \sum_{k=1}^{K} \int_{\theta_k-\delta}^{\theta_k+\delta} \Phi_{\theta_k}(\theta) b^{(\lambda)}(\theta) e^{-in\theta} d\theta + Q_1(\lambda, n), \tag{4.1}$$

where $Q_1(\lambda,n) = o(1/n^\infty)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$.

Proof. Using the partition $\mathcal{P}$, we may write $I = I_1 + \cdots + I_K + I^*$ where

$$I_k := \frac{1}{2\pi} \int_{\theta_k-\delta}^{\theta_k+\delta} \Phi_{\theta_k}(\theta) \hat{b}^{(\lambda)}(\theta) e^{-in\theta} d\theta$$

for $k = 1, \ldots, K$ and

$$I^* := \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi^*(\theta) \hat{b}^{(\lambda)}(\theta) e^{-in\theta} d\theta.$$

Taking $v(\theta) := \Phi^*(\theta)\hat{b}^{(\lambda)}(\theta)$, $\alpha := \theta_1$, and $\beta := 2\pi + \theta_1$ in Theorem 4.1 we easily get $I^* = o(1/n^\infty)$ as $n \to \infty$, uniformly with respect to $\lambda \in a(W) \setminus W_0$.

Using (3.1), we arrive at $I_k = I_{k1} - I_{k2}$ where

$$I_{k1} := \frac{1}{2\pi} \int_{\theta_k-\delta}^{\theta_k+\delta} \Phi_{\theta_k}(\theta) b^{(\lambda)}(\theta) e^{-in\theta} d\theta \tag{4.2}$$

and

$$I_{k2} := \frac{1}{2\pi} \int_{\theta_k-\delta}^{\theta_k+\delta} \frac{\Phi_{\theta_k}(\theta) e^{-in\theta}}{t_\lambda a'(t_\lambda)\big(e^{i\theta} - t_\lambda\big)} d\theta.$$

Finally, letting $v(\theta) := \Phi_{\theta_k}(\theta)/\big(t_\lambda a'(t_\lambda)(e^{i\theta} - t_\lambda)\big)$, $\alpha := \theta_k - \delta$, and $\beta := \theta_k + \delta$ in Theorem 4.1 we easily obtain $I_{k2} = o(1/n^\infty)$ as $n \to \infty$, uniformly with respect to λ in $a(W) \setminus W_0$. □

Expression (4.1) says that the value of I basically depends on the integrand $b^{(\lambda)}(\theta)e^{-in\theta}$ at the singularity arguments θ_k. As we can take δ as small as we desire, we may assume that in every integral of the sum of (4.1) the variable θ is arbitrarily close to θ_k. Keeping this idea in mind, we will develop an asymptotic expansion for $b^{(\lambda)}$. For future reference, we rewrite (4.1) as

$$I = \sum_{k=1}^{K} I_{k1} + Q_1(\lambda, n), \tag{4.3}$$

where $Q_1(\lambda,n) = o(1/n^\infty)$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$. Writing $h(\theta)$ instead of $h(e^{i\theta})$, we obtain the following lemma.

Lemma 4.4. *For every $k \in \{1, \ldots, K\}$ and every sufficiently small positive δ,*

$$I_{k1} = \frac{-1}{2\pi\lambda} \sum_{s=0}^{\infty} \frac{1}{\lambda^s} \int_{\theta_k-\delta}^{\theta_k+\delta} \frac{\Phi_{\theta_k}(\theta) h^s(\theta) e^{-in\theta}}{e^{i\theta(s+1)}} d\theta. \tag{4.4}$$

Proof. Note that

$$b^{(\lambda)}(\theta) = \frac{1}{h(\theta)-\lambda e^{i\theta}} = \frac{-1}{\lambda e^{i\theta}} \cdot \frac{1}{1-\lambda^{-1}e^{-i\theta}h(\theta)}.$$

Let $k \in \{1,\dots,K\}$. As $|h(\theta)| \to 0$ when $|\theta-\theta_k| \to 0$, there is a small positive constant δ_k such that $|\lambda^{-1}e^{-i\theta}h(\theta)| < 1$ for every $|\theta-\theta_k| < \delta_k$. Let $\delta = \min_{1\le k\le K}\{\delta_k\}$. Thus,

$$b^{(\lambda)}(\theta) = \frac{-1}{\lambda e^{i\theta}} \sum_{s=0}^{\infty} \left(\lambda^{-1}e^{-i\theta}h(\theta)\right)^s = -\sum_{s=0}^{\infty} \frac{h^s(\theta)}{\lambda^{s+1}e^{i\theta(s+1)}} \tag{4.5}$$

for every $k \in \{1,\dots,K\}$ and every $|\theta-\theta_k| < \delta$. Finally, inserting (4.5) in (4.2) finishes the proof. □

We will use the notation

$$I_{k1s} := \frac{-1}{2\pi\lambda^{s+1}} \int_{\theta_k-\delta}^{\theta_k+\delta} \frac{\Phi_{\theta_k}(\theta)h^s(\theta)e^{-in\theta}}{e^{i\theta(s+1)}}d\theta. \tag{4.6}$$

Once more, taking $v(\theta) := -\Phi_{\theta_k}(\theta)/\left(2\pi\lambda e^{i\theta}\right)$, $\alpha := \theta_k - \delta$, and $\beta := \theta_k + \delta$ in Theorem 4.1 we easily obtain $I_{k1s}|_{s=0} = o(1/n^\infty)$ as $n \to \infty$, uniformly with respect to $\lambda \in a(W)\setminus W_0$. With the previous notation, we can rewrite (4.4) as

$$I_{k1} = \sum_{s=1}^{\infty} I_{k1s} + Q_2(k,\lambda,n),$$

where $Q_2(k,\lambda,n) = o(1/n^\infty)$ as $n \to \infty$, uniformly with respect to $\lambda \in a(W)\setminus W_0$. Now we use Theorem 4.2 to express I_{k1s} asymptotically. We recall that $h(t) = f(t)\prod_{k=1}^{K}(1-t/t_k)^{\alpha_k}$, where $t_k = e^{i\theta_k}$, the arguments θ_k are all different, and the exponents α_k are positive reals but not integers, with $\alpha_1 = \min_{1\le k\le K}\{\alpha_k\}$.

Lemma 4.5. *Let f be a function with property 2 and μ be any positive real number. Then, for $k \in \{1,\dots,K\}$,*

$$I_{k1} = \sum_{(\ell,s)\in\mathcal{L}_\mu^*} \frac{A_{k,\ell,s}}{\lambda^{s+1}t_k^n n^{\alpha_k s+\ell+1}} + Q_7(k,\lambda,n), \tag{4.7}$$

where $\mathcal{L}_\mu^*$ is the collection of all pairs (ℓ,s) such that $\ell \in \{0,1,\dots\}$, $s \in \{1,2,\dots\}$, and $\alpha_k s+\ell+1 < \mu$;

$$A_{k,\ell,s} = \frac{\sin(\alpha_k\pi s)\Gamma(\alpha_k s+\ell+1)}{i^\ell \pi t_k^{s+1}\ell!}\left[\frac{f^s(t_k e^{i\theta})g^{\alpha_k s}(\theta)\prod_{j\ne k}(1-e^{i\theta}t_k/t_j)^{\alpha_j s}}{e^{i\theta(s+1)}}\right]^{(\ell)}_{\theta=0},$$

$g(\theta) = (e^{i\theta}-1)/(i\theta)$, and $Q_7(k,\lambda,n) = O(1/n^\mu)$ as $n \to \infty$, uniformly with respect to $\lambda \in a(W)\setminus W_0$.

Proof. Changing θ to $\theta+\theta_k$ in (4.6), we obtain

$$I_{k1s} = \frac{-1}{2\pi\lambda^{s+1}} \int_{-\delta}^{\delta} \frac{\Phi_0(\theta)f^s(t_k e^{i\theta})\left(1-e^{i\theta}\right)^{\alpha_k s}\prod_{j\ne k}\left(1-e^{i\theta}t_k/t_j\right)^{\alpha_j s}e^{-in\theta}}{e^{i\theta(s+1)}t_k^{n+s+1}}d\theta.$$

It is easy to verify that $1-e^{i\theta} = -i\theta g(\theta)$, where $g(\theta) := 1+i\theta/2+(i\theta)^2/6+O(\theta^3)$ as $\theta \to 0$. Thus, we can write $I_{k1s} = \int_{-\delta}^{\delta}\theta^{\alpha_k s}v(\theta)e^{-in\theta}d\theta$, where

$$v(\theta) := \frac{-(-i)^{\alpha_k s}\Phi_0(\theta)f^s(t_k e^{i\theta})g^{\alpha_k s}(\theta)\prod_{j\ne k}\left(1-e^{i\theta}t_k/t_j\right)^{\alpha_j s}}{2\pi\lambda^{s+1}e^{i\theta(s+1)}t_k^{n+s+1}},$$

the branch of the power $\alpha_k s$ being the one corresponding to the argument in $(-\pi,\pi]$. Note that for every sufficiently small positive δ we have $g \in C^\infty[-\delta,\delta]$ and $g(0)=1$. Clearly,

$$\begin{aligned} I_{k1s} &= \int_{-\delta}^{0} \theta^{\alpha_k s} v(\theta) e^{-in\theta} d\theta + \int_{0}^{\delta} \theta^{\alpha_k s} v(\theta) e^{-in\theta} d\theta \\ &= \int_{0}^{\delta} (-\theta)^{\alpha_k s} v(-\theta) e^{in\theta} d\theta + \int_{0}^{\delta} \theta^{\alpha_k s} v(\theta) e^{-in\theta} d\theta = I_{k1s1} + I_{k1s2}, \end{aligned} \tag{4.8}$$

where

$$I_{k1s1} := (-1)^{\alpha_k s} \int_0^\delta \theta^{\alpha_k s} v(-\theta) e^{in\theta} d\theta, \quad I_{k1s2} := \int_0^\delta \theta^{\alpha_k s} v(\theta) e^{-in\theta} d\theta.$$

Note that $v(\pm\theta) \in C^\infty[0,\delta]$ and $v^{(s)}(\pm\delta)=0$ for all $s \geq 0$ because $\Phi_0 \equiv 0$ in a small neighborhood of $\pm\delta$. Applying Theorem 4.2 to I_{k1s1} and $\overline{I_{k1s2}}$, we obtain

$$\begin{aligned} I_{k1s1} &= \sum_{\ell=0}^{L-1} \frac{(-1)^{\alpha_k s+\ell} v^{(\ell)}(0) \Gamma(\alpha_k s+\ell+1) i^{\alpha_k s+\ell+1}}{n^{\alpha_k s+\ell+1} \ell!} + Q_3(s,k,L,\lambda,n), \\ I_{k1s2} &= \sum_{\ell=0}^{L-1} \frac{v^{(\ell)}(0) \Gamma(\alpha_k s+\ell+1) i^{-\alpha_k s-\ell-1}}{n^{\alpha_k s+\ell+1} \ell!} + Q_4(s,k,L,\lambda,n), \end{aligned} \tag{4.9}$$

for every $L \in \mathbb{N}$, where Q_3 and Q_4 are $O(1/n^{\alpha_k s+L+1})$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$. Substitution of (4.9) in (4.8) yields

$$\begin{aligned} I_{k1s} = &\sum_{\ell=0}^{L-1} \frac{v^{(\ell)}(0) \Gamma(\alpha_k s+\ell+1)}{n^{\alpha_k s+\ell+1} \ell!} \left(i^{-\alpha_k s-\ell-1} + (-1)^{\alpha_k s+\ell} i^{\alpha_k s+\ell+1} \right) \\ &+ Q_5(s,k,L,\lambda,n), \end{aligned}$$

for every $L \in \mathbb{N}$, where $Q_5(s,k,L,\lambda,n) = O(1/n^{\alpha_k s+L+1})$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$. At this point, one could be tempted to write

$$I_{k1} = \sum_{s=1}^{\infty} \left(\sum_{\ell=0}^{L-1} \frac{A_{k,\ell,s}}{\lambda^{s+1} t_k^n n^{\alpha_k s+\ell+1}} + Q_5(s,k,L,\lambda,n) \right) + Q_2(k,\lambda,n) \text{ as } n \to \infty, \tag{4.10}$$

where $A_{k,\ell,s}$ equals

$$\frac{\sin(\alpha_k \pi s) \Gamma(\alpha_k s+\ell+1)}{i^\ell \pi t_k^{s+1} \ell!} \left[\frac{\Phi_0(\theta) f^s(t_k e^{i\theta}) g^{\alpha_k s}(\theta) \prod_{j\neq k} (1-e^{i\theta} t_k/t_j)^{\alpha_j s}}{e^{i\theta(s+1)}} \right]^{(\ell)}_{\theta=0}.$$

Note that we can drop the factor $\Phi_0(\theta)$ above because $\Phi_0 \equiv 1$ in a neighborhood of $\theta = 0$. However, representation (4.10) does not permit us to get an appropriate bound for the remainder of I_{k1}. We therefore tackle the problem as follows. First notice that

$$\begin{aligned} h(\theta+\theta_k) &= f(\theta+\theta_k) \prod_{j=1}^{K} (1-e^{i\theta} t_k/t_j)^{\alpha_j} \\ &= (1-e^{i\theta})^{\alpha_k} f(\theta+\theta_k) \prod_{j\neq k} (1-e^{i\theta} t_k/t_j)^{\alpha_j} = O(\theta^{\alpha_k}) \text{ as } \theta \to 0. \end{aligned}$$

Thus, from (4.5) we obtain

$$b^{(\lambda)}(\theta+\theta_k) = -\sum_{s=0}^{S-1} \frac{h^s(\theta+\theta_k)}{\lambda^{s+1} e^{i(\theta+\theta_k)(s+1)}} + f_{k,S}^{(\lambda)}(\theta) \tag{4.11}$$

for every $S \in \mathbb{N}$ and every $k \in \{1,\dots,K\}$. Here $f_{k,S}^{(\lambda)}(\theta) = O(\theta^{\alpha_k S})$ as $\theta \to 0$, uniformly in $\lambda \in a(W) \setminus W_0$. Inserting (4.11) in (4.2) and (4.3) we obtain

$$\begin{aligned} I_{k1} &= \sum_{s=1}^{S-1} I_{k1s} + \frac{1}{2\pi}\int_{-\delta}^{\delta} \Phi_0(\theta) f_{k,S}^{(\lambda)}(\theta) e^{-in\theta} d\theta + Q_2(k,\lambda,n) \\ &= \sum_{s=1}^{S-1}\sum_{\ell=0}^{L-1} \frac{A_{k,\ell,s}}{\lambda^{s+1} t_k^n n^{\alpha_k s+\ell+1}} + \sum_{s=1}^{S-1} Q_5(s,k,L,\lambda,n) \\ &\quad + \frac{1}{2\pi}\int_{-\delta}^{\delta} \Phi_0(\theta) f_{k,S}^{(\lambda)}(\theta) e^{-in\theta} d\theta + Q_2(k,\lambda,n) \end{aligned} \tag{4.12}$$

for every $L,S \in \mathbb{N}$. The function $\Phi_0(\theta) f_{k,S}^{(\lambda)}(\theta)$ belongs to $C^{[\alpha_k S]}[-\delta,\delta]$ and thus by Theorem 4.1, the integral on the right side of (4.12) is $o(1/n^{[\alpha_k S]})$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$.

Fix $S \in \mathbb{N}$ such that $[\alpha_1 S] > \mu$. Then, the integral on the right side of (4.12) is $o(1/n^{\mu})$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$ for every $k \in \{1,\dots,K\}$.

Now fix $L \in \mathbb{N}$ such that $\alpha_1 + L + 1 > \mu$. Thus, $Q_5(s,k,L,\lambda,n) = O(1/n^{\mu})$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$ for every $k \in \{1,\dots,K\}$. Therefore, the finite sum $\sum_{s=1}^{S-1} Q_5(s,k,L,\lambda,n)$ is $O(1/n^{\mu})$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$ for every $k \in \{1,\dots,K\}$.

In summary,

$$I_{k1} = \sum_{s=1}^{S-1}\sum_{\ell=0}^{L-1} \frac{A_{k,\ell,s}}{\lambda^{s+1} t_k^n n^{\alpha_k s+\ell+1}} + Q_6(k,\lambda,n),$$

where $Q_6(S,k,L,\lambda,n) = O(1/n^{\mu})$ as $n \to \infty$, uniformly in $\lambda \in a(W) \setminus W_0$ for every $k \in \{1,\dots,K\}$. Finally, avoiding the unnecessary terms of the sum we finish the proof. □

Proof of Theorem 1.1. Combine (2.1), (3.2), (4.3), and (4.7). □

5 Individual eigenvalues

In order to find the eigenvalues of the matrices $T_n(a)$, we need to solve the equations $D_n(a-\lambda) = 0$. We start this Section by locating the zeros of $D_n(a-\lambda)$.

Let W_0 be a small open neighborhood of zero in $\mathbb{C}$ and $\omega_n := \exp(-2\pi i/n)$. Let

$$\mathcal{J}_n := \left\{ j \in \{0,\dots,n-1\} : a(\omega_n^j) \notin W_0 \right\}. \tag{5.1}$$

Recall that $\lambda = a(t_\lambda)$. Take an integer $j \in \mathcal{J}_n$. Using the representations

$$\frac{1}{t_\lambda^2 a'(t_\lambda)} = \frac{1}{\omega_n^{2j} a'(\omega_n^j)} + O(|t_\lambda - \omega_n^j|), \quad \frac{1}{a^2(t_\lambda)} = \frac{1}{a^2(\omega_n^j)} + O(|t_\lambda - \omega_n^j|),$$

where t_λ belongs to a small neighborhood of ω_n^j, we see that the determinant $D_n(a-\lambda)$ in (1.1) equals $(-h_0)^{n+1}$ times

$$\mathcal{T}_1 - \mathcal{T}_2 + O\left(\left|\frac{t_\lambda - \omega_n^j}{t_\lambda^n}\right|\right) + O\left(\frac{|t_\lambda - \omega_n^j|}{n^{\alpha_1+1}}\right) + R_1(\lambda,n), \tag{5.2}$$

where t_λ belongs to a small neighborhood of ω_n^j,

$$\mathcal{T}_1 := \frac{1}{t_\lambda^n \omega_n^{2j} a'(\omega_n^j)}, \quad \mathcal{T}_2 := \sum_{(k,\ell,s)\in\mathcal{L}_\mu} \frac{A_{k,\ell,s}}{a^{s+1}(\omega_n^j) t_k^n n^{\alpha_k s+\ell+1}} = \frac{A_{1,0,1}\left(1+Q_8(\lambda,n)\right)}{a^2(\omega_n^j) n^{\alpha_1+1}}$$

with $Q_8(\lambda,n) = O(1/n^{\zeta})$ as $n \to \infty$, uniformly with respect to $\lambda \in a(W) \setminus W_0$. Here $\mathcal{L}_\mu$, $A_{k,\ell,s}$, and ζ are as in Theorem 1.1. Expression (5.2) makes sense only when t_λ is sufficiently close to ω_n^j and thus it is necessary to know whether there is a zero of $D_n(a-\lambda)$ close to ω_n^j. Let $t_\lambda := \rho e^{i\phi}$. It is easy to verify that $\mathcal{T}_1 - \mathcal{T}_2 = 0$ if and only if

$$\rho = \left(\frac{|a(\omega_n^j)|^2 \left|1 + Q_9(n)\right| n^{\alpha_1+1}}{|A_{1,0,1} a'(\omega_n^j)|} \right)^{1/n} \tag{5.3}$$

and

$$\phi = \phi_s = \frac{1}{n} \arg\left(\frac{a^2(\omega_n^j)\big(1+Q_9(n)\big)}{A_{1,0,1}\,\omega_n^{2j}\, a'(\omega_n^j)} \right) - \frac{2\pi s}{n}$$

where $s \in \{0,\ldots,n-1\}$ and $Q_9(\lambda,n) = O(1/n^{\zeta})$ as $n \to \infty$, uniformly with respect to $\lambda \in a(W) \setminus W_0$. When n tends to infinity, (5.3) shows that ρ remains greater than 1 and $\rho \to 1$. The function $\mathcal{T}_1 - \mathcal{T}_2$ has n zeros with respect to $\lambda \in \mathcal{D}(a)$ given by

$$a\big(\rho e^{i\phi_0}\big), \quad \ldots, \quad a\big(\rho e^{i\phi_{n-1}}\big).$$

As Lemma 3.1 establishes a 1-1 correspondence between λ and t_λ, the function $D_n(a-\lambda)$ is analytic with respect to $\lambda \in a(W) \setminus W_0$, that is, analytic with respect to $t_\lambda \in W \setminus a^{-1}(W_0)$. We can therefore suppose that $\mathcal{T}_1 - \mathcal{T}_2$ has n zeros with respect to t_λ in the exterior of $\overline{\mathbb{D}}$ given by

$$z_0 := \rho e^{i\phi_0}, \quad \ldots, \quad z_{n-1} := \rho e^{i\phi_{n-1}}.$$

We take the function "arg" in the interval $(-\pi,\pi]$. Thus, $z_j = e^{i\phi_j}$ is the nearest zero to ω_n^j. Consider the open neighborhood E_j of z_j sketched in Figure 4.

The boundary of E_j is $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$. We have chosen radial segments Γ_2 and Γ_4 so that their length is $1/n^{\varepsilon}$ with $\varepsilon \in (0, \min\{1, \alpha_1, \gamma - \alpha_1\})$ and $\gamma = \min\{\alpha_j : \alpha_j > \alpha_1\}$ and all the points in Γ_2 have the common argument $(\phi_{j+1} + \phi_j)/2$, while all the points in Γ_4 have the common argument $(\phi_{j-1} + \phi_j)/2$. As we can see in Figure 4, these points run from the unit circle $\mathbb{T}$ to $(1+1/n^{\varepsilon})\mathbb{T}$. Note also that $\Gamma_1 \subset (1+1/n^{\varepsilon})\mathbb{T}$ and $\Gamma_3 \in \mathbb{T}$. Recall $\mathcal{J}_n$ from (5.1). We put $\operatorname{diam}(E_j) := \sup\{|z_1 - z_2| : z_1, z_2 \in E_j\}$.

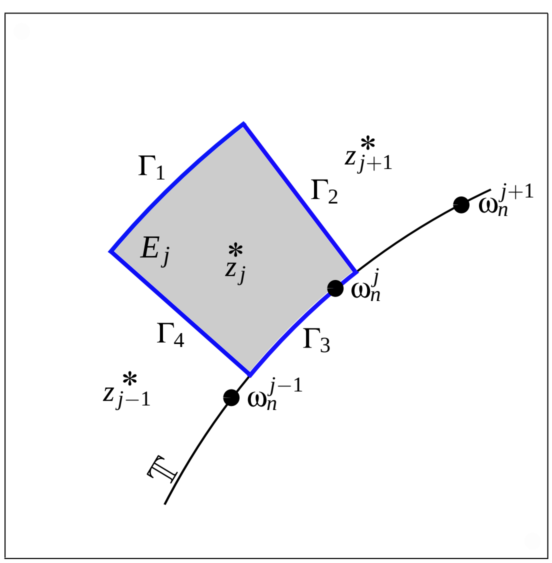

Figure 4. The neighborhood E_j of z_j in the complex plane.

Theorem 5.1. *Suppose $a(t) = t^{-1}h(t)$ is a symbol with properties 1 to 4. Let $\varepsilon \in (0, \min\{1, \alpha_1, \gamma - \alpha_1\})$ be a constant. Then, there is a family of sets $\{E_j\}_{j\in\mathcal{J}_n}$ in $\mathbb{C}$ such that*

1. *$\{E_j\}_{j\in\mathcal{J}_n}$ is a family of pairwise disjoint open sets,*
2. *$\operatorname{diam}(E_j) \le 2/n^{\varepsilon}$,*
3. *$\omega_n^j \in \partial E_j$,*
4. *$D_n\big(a - a(t_\lambda)\big) = D_n(a-\lambda)$ has exactly one zero in each E_j.*

Proof. Assertions 1, 2, and 3 can be deduced from the above construction. We prove assertion 4 by studying the behavior of $|D_n(a-\lambda)|$ in dependence on $t_\lambda \in \Gamma$. For $t_\lambda \in \Gamma_1$ we have, as $n \to \infty$,

$$|\mathcal{T}_1|_{\Gamma_1} = \frac{1}{|a'(\omega_n^j)|}\left(1+\frac{1}{n^{\varepsilon}}\right)^{-n} = \frac{\exp(-n^{1-\varepsilon})}{|a'(\omega_n^j)|} + O\left(\frac{\exp(-n^{1-\varepsilon})}{n^{2\varepsilon-1}}\right),$$

$$|\mathcal{T}_2|_{\Gamma_1} = \frac{1}{n^{\alpha_1+1}}\left|\frac{A_{1,0,1}\big(1+Q_8(n)\big)}{a^2(\omega_n^j)}\right|,$$

$$\left|O\left(\left|\frac{t_\lambda-\omega_n^j}{t_\lambda^n}\right|\right)\right|_{\Gamma_1} = O\left(\frac{\exp(-n^{1-\varepsilon})}{n^{\varepsilon}}\right), \quad \left|O\left(\frac{|t_\lambda-\omega_n^j|}{n^{\alpha_1+1}}\right)\right|_{\Gamma_1} = O\left(\frac{1}{n^{\alpha_1+\varepsilon+1}}\right),$$

and $\big|R_1(n,t_\lambda)\big|_{\Gamma_1} = O(1/n^{\mu})$. When n goes to infinity, the absolute value of $\mathcal{T}_2$ decreases at polynomial speed over Γ_1, while the absolute values of the remaining terms in (5.2) are smaller over Γ_1. Thus,

$$\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_1} = \frac{1}{n^{\alpha_1+1}}\left|\frac{A_{1,0,1}}{a^2(\omega_n^j)}\right| + O\left(\frac{1}{n^{\alpha_1+\varepsilon+1}}\right) \text{ as } n\to\infty.$$

For $t_\lambda \in \Gamma_3$, as $n \to \infty$, we get

$$|\mathcal{T}_1|_{\Gamma_3} = \frac{1}{|a'(\omega_n^j)|}, \quad |\mathcal{T}_2|_{\Gamma_3} = \frac{1}{n^{\alpha_1+1}}\left|\frac{A_{1,0,1}\big(1+Q_8(n)\big)}{a^2(\omega_n^j)}\right|,$$

$$\left|O\left(\left|\frac{t_\lambda-\omega_n^j}{t_\lambda^n}\right|\right)\right|_{\Gamma_3} = O\left(\frac{1}{n}\right), \quad \left|O\left(\frac{|t_\lambda-\omega_n^j|}{n^{\alpha_1+1}}\right)\right|_{\Gamma_3} = O\left(\frac{1}{n^{\alpha_1+2}}\right),$$

and $\big|R_1(n,t_\lambda)\big|_{\Gamma_3} = O(1/n^{\mu})$. When n goes to infinity, the modulus of $\mathcal{T}_1$ remains constant over Γ_3, while the moduli of the remaining terms in (5.2) are smaller there. Consequently,

$$\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_3} = \frac{1}{|a'(\omega_n^j)|} + O\left(\frac{1}{n}\right) \text{ as } n\to\infty.$$

As for the radial segments Γ_2 and Γ_4, we start by showing that $\mathcal{T}_1$ and $-\mathcal{T}_2$ have the same argument there. Since z_j is a zero of $\mathcal{T}_1 - \mathcal{T}_2$, we deduce that

$$\arg\left(\frac{1}{z_j^n \omega_n^{2j} a'(\omega_n^j)}\right) = \arg\left(\frac{A_{1,0,1}\big(1+Q_8(n)\big)}{a^2(\omega_n^j) n^{\alpha_1+1}}\right)$$

as $n\to\infty$ and thus

$$-n\phi_j+\arg\left(\frac{1}{\omega_n^{2j}a'(\omega_n^j)}\right)=\arg\left(\frac{A_{1,0,1}\left(1+Q_8(n)\right)}{a^2(\omega_n^j)}\right). \tag{5.4}$$

For $t_\lambda\in\Gamma_2$ we have

$$\begin{aligned}\arg(\mathcal{T}_1)=\arg\left(\frac{1}{t_\lambda^n\omega_n^{2j}a'(\omega_n^j)}\right)&=-\frac{n}{2}(\phi_{j-1}+\phi_j)+\arg\left(\frac{1}{\omega_n^{2j}a'(\omega_n^j)}\right)\\&=\frac{n}{2}(\phi_j-\phi_{j-1})+\arg\left(\frac{A_{1,0,1}\left(1+Q_8(n)\right)}{a^2(\omega_n^j)}\right)\\&=\pi+\arg\left(\frac{A_{1,0,1}\left(1+Q_8(n)\right)}{a^2(\omega_n^j)}\right)=\arg(-\mathcal{T}_2).\end{aligned}$$

Here the third line is due to (5.4). In addition, as $n\to\infty$,

$$\left|O\left(\left|\frac{t_\lambda-\omega_n^j}{t_\lambda^n}\right|\right)\right|_{\Gamma_2}=O\left(\frac{1}{n^{\varepsilon}|t_\lambda|^n}\right),\quad\left|O\left(\frac{|t_\lambda-\omega_n^j|}{n^{\alpha_1+1}}\right)\right|_{\Gamma_2}=O\left(\frac{1}{n^{\alpha_1+\varepsilon+1}}\right),$$

and $\left|R_1(n,t_\lambda)\right|_{\Gamma_2}=O(1/n^{\mu})$. Furthermore,

$$\left|\frac{D_n(a-\lambda)}{h_0^{n+1}}\right|_{\Gamma_2}=\frac{1}{|t_\lambda^n a'(\omega_n^j)|}+O\left(\frac{1}{n^{\varepsilon}|t_\lambda|^n}\right)+\frac{1}{n^{\alpha_1+1}}\left|\frac{A_{1,0,1}}{a^2(\omega_n^j)}\right|+O\left(\frac{1}{n^{\alpha_1+\varepsilon+1}}\right)$$

over Γ_2 as $n\to\infty$. The situation is similar for the segment Γ_4.

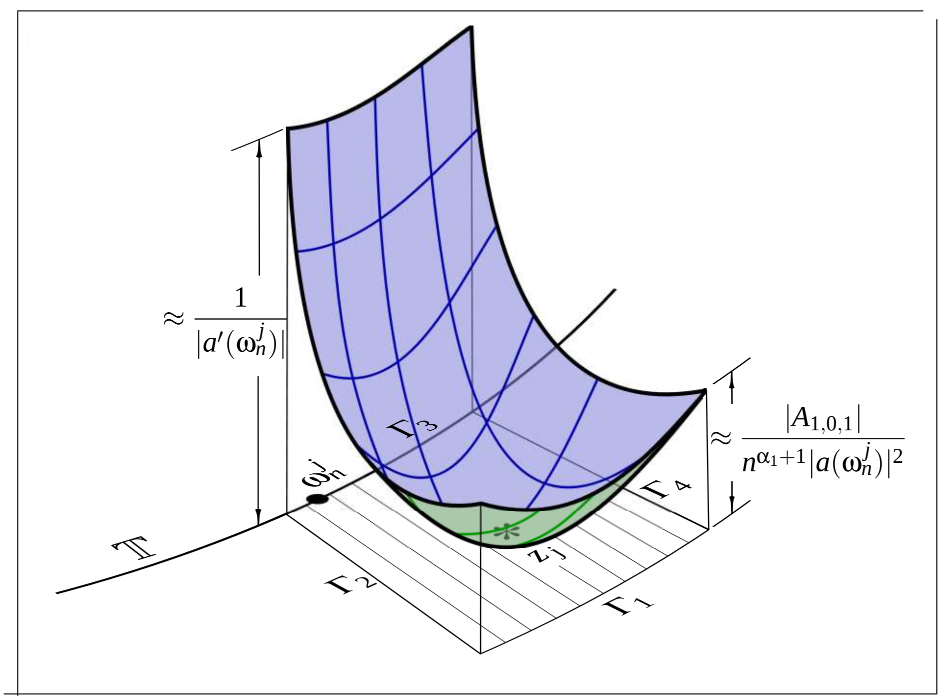

Figure 5. The absolute value of $D_n(a-\lambda)/h_0^{n+1}$ over E_j.

Figure 5 resumes our analysis of $|D_n(a-\lambda)/h_0^{n+1}|$. From the previous study of $|D_n(a-\lambda)|$ over Γ we infer that for every sufficiently large n we have

$$|\mathcal{T}_1-\mathcal{T}_2|_\Gamma \geq \frac{1}{2n^{\alpha_1+1}}\left|\frac{A_{1,0,1}}{a^2(\omega_n^j)}\right|$$

and

$$\left|\mathcal{O}\left(\left|\frac{t_\lambda-\omega_n^j}{t_\lambda^n}\right|\right)+\mathcal{O}\left(\frac{|t_\lambda-\omega_n^j|}{n^{\alpha_1+1}}\right)+R_1(n,t_\lambda)\right|_\Gamma \leq \mathcal{O}\left(\frac{1}{n^{\alpha_1+\varepsilon+1}}\right).$$

Hence, by Rouché's theorem, $D_n(a-\lambda)/(-h_0)^{n+1}$ and $\mathcal{T}_1-\mathcal{T}_2$ have the same number of zeros in E_j, that is, a unique zero. □

As a consequence of Theorem 5.1, we can iterate the variable t_λ in the equation $D_n(a-\lambda)=0$, where $D_n(a-\lambda)$ is given by (1.1). In this fashion we find the unique eigenvalue of $T_n(a)$ which is located close to $a(\omega_n^j)$.

Proof of Theorem 1.2. The equation $D_n(a-\lambda)=0$ with $D_n(a-\lambda)$ given by (1.1) is equivalent to the equation

$$t_\lambda^{-n}=\frac{A_{1,0,1}t_\lambda^2 a'(t_\lambda)}{a^2(t_\lambda)n^{\alpha_1+1}}\left(1+\frac{1}{A_{1,0,1}}\sum_{\substack{(k,\ell,s)\in\mathcal{L}_\mu\\(k,\ell,s)\neq(1,0,1)}}\frac{A_{k,\ell,s}}{a^{s-1}(t_\lambda)t_k^n n^{\alpha_k s+\ell-\alpha_1}}+Q_{10}(n,t_\lambda)\right), \tag{5.5}$$

where $Q_{10}(n,t_\lambda)=\mathcal{O}(1/n^{\mu-\alpha_1-1})$ as $n\to\infty$, uniformly with respect to $t_\lambda\in W\setminus a^{-1}(W_0)$. Recall from Theorem 1.1 that $\gamma=\min\{\alpha_j\colon \alpha_j>\alpha_1\}$ and $\zeta=\min\{1,\alpha_1,\gamma-\alpha_1\}$. As μ is any real positive number, we can develop (5.5) with an arbitrary error bound, but to make our calculations reasonable and readable, we limit ourselves to $\mu=2\zeta+\alpha_1+1$. Equation (5.5) is an implicit expression for t_λ. We manipulate it to obtain a few asymptotic terms for t_λ. Remember that λ belongs to $\mathcal{D}(a)\setminus W_0$; see Figure 3. We can choose W so thin that $\lambda=a(t_\lambda)$, $a'(t_\lambda)$, and t_λ are bounded and not too close to zero. After taking the nth root for the main branch specified by the argument in $(-\pi,\pi]$ and expanding in (5.5),

$$\begin{aligned} t_{\lambda_{j,n}} =& \omega_n^j n^{(\alpha_1+1)/n}\left(1+\sum_{m=1}^{[1+2\zeta]}\log^m\left(\frac{a^2(t_{\lambda_{j,n}})}{A_{1,0,1}t_{\lambda_{j,n}}^2 a'(t_{\lambda_{j,n}})}\right)\frac{1}{m!\,n^m}+Q_{11}(j,n)\right)\\ &\times\left(1-\frac{1}{A_{1,0,1}}\sum_{\substack{(k,\ell,s)\in\mathcal{L}_\mu\\(k,\ell,s)\neq(1,0,1)}}\frac{A_{k,\ell,s}}{a^{s-1}(t_{\lambda_{j,n}})t_k^n n^{\alpha_k s+\ell-\alpha_1+1}}+Q_{12}(j,n)\right), \end{aligned} \tag{5.6}$$

where Q_{11} and Q_{12} are $\mathcal{O}(1/n^{2\zeta+1})$ as $n\to\infty$, uniformly with respect to $j\in\mathcal{J}_n$. After multiplying in (5.6) we obtain

$$\begin{aligned} t_{\lambda_{j,n}} =& \omega_n^j n^{(\alpha_1+1)/n}\left(1+\sum_{m=1}^{[1+2\zeta]}\log^m\left(\frac{a^2(t_{\lambda_{j,n}})}{A_{1,0,1}t_{\lambda_{j,n}}^2 a'(t_{\lambda_{j,n}})}\right)\frac{1}{m!\,n^m}\right.\\ &\left.-\frac{1}{A_{1,0,1}}\sum_{\substack{(k,\ell,s)\in\mathcal{L}_\mu\\(k,\ell,s)\neq(1,0,1)}}\frac{A_{k,\ell,s}}{a^{s-1}(t_{\lambda_{j,n}})t_k^n n^{\alpha_k s+\ell-\alpha_1+1}}+Q_{13}(j,n)\right), \end{aligned} \tag{5.7}$$

where $Q_{13}(n,t_\lambda)=O(1/n^{2\zeta+1})$ as $n\to\infty$, uniformly with respect to $t_\lambda\in W\setminus a^{-1}(W_0)$. Note that, as $n\to\infty$,

$$n^{(\alpha_1+1)/n}=\exp\left((\alpha_1+1)\frac{\log n}{n}\right)=1+(\alpha_1+1)\frac{\log n}{n}+O\left(\frac{\log^2 n}{n^2}\right). \tag{5.8}$$

Thus, our first approximation for $t_{\lambda_{j,n}}$ is

$$t_{\lambda_{j,n}}=\omega_n^j+Q_{14}(j,n),$$

where $Q_{14}(j,n)=O(\log n/n)$ as $n\to\infty$, uniformly with respect to $j\in\mathcal{J}_n$. Replacing $t_{\lambda_{j,n}}$ by this approximation in (5.7) we obtain

$$\begin{aligned}t_{\lambda_{j,n}}=&\omega_n^j n^{(\alpha_1+1)/n}\left(1+\sum_{m=1}^{[1+2\zeta]}\log^m\left(\frac{a^2(\omega_n^j)}{A_{1,0,1}\omega_n^{2j}a'(\omega_n^j)}\right)\frac{1}{m!\,n^m}\right.\\&\left.-\frac{1}{A_{1,0,1}}\sum_{\substack{(k,\ell,s)\in\mathcal{L}_\mu\\(k,\ell,s)\neq(1,0,1)}}\frac{A_{k,\ell,s}}{t_k^n a^{s-1}(\omega_n^j)n^{\alpha_k s+\ell-\alpha_1+1}}+R_2(j,n)\right),\end{aligned}$$

where $R_2(j,n)=O(1/n^{2\zeta+1})+O(\log n/n^2)$ as $n\to\infty$, uniformly with respect to $j\in\mathcal{J}_n$. □

Proof of Theorem 1.3. Inserting (5.8) in (1.2) we obtain

$$\begin{aligned}t_{\lambda_{j,n}}=&\omega_n^j\left(1+(\alpha_1+1)\frac{\log n}{n}+\sum_{m=1}^{[1+2\zeta]}\log^m\left(\frac{a^2(\omega_n^j)}{A_{1,0,1}\omega_n^{2j}a'(\omega_n^j)}\right)\frac{1}{m!\,n^m}\right.\\&\left.-\frac{1}{A_{1,0,1}}\sum_{\substack{(k,\ell,s)\in\mathcal{L}_\mu\\(k,\ell,s)\neq(1,0,1)}}\frac{A_{k,\ell,s}}{t_k^n a^{s-1}(\omega_n^j)n^{\alpha_k s+\ell-\alpha_1+1}}+Q_{15}(j,n)\right),\end{aligned} \tag{5.9}$$

where $Q_{15}(j,n)=O(1/n^{2\zeta+1})+O(\log^2 n/n^2)$ as $n\to\infty$, uniformly with respect to $j\in\mathcal{J}_n$.

Since the symbol a is analytic in a small neighborhood of each $t_{\lambda_{j,n}}$, we have $\lambda_{j,n}=a(t_{\lambda_{j,n}})=a(\omega_n^j+z)=a(\omega_n^j)+a'(\omega_n^j)z+O(|z|^2)$. Thus, applying the symbol a to (5.9), we get

$$\begin{aligned}\lambda_{j,n}=&a(\omega_n^j)+(\alpha_1+1)\omega_n^j a'(\omega_n^j)\frac{\log n}{n}\\&+\omega_n^j a'(\omega_n^j)\sum_{m=1}^{[1+2\zeta]}\log^m\left(\frac{a^2(\omega_n^j)}{A_{1,0,1}\omega_n^{2j}a'(\omega_n^j)}\right)\frac{1}{m!\,n^m}\\&-\frac{\omega_n^j a'(\omega_n^j)}{A_{1,0,1}}\sum_{\substack{(k,\ell,s)\in\mathcal{L}_\mu\\(k,\ell,s)\neq(1,0,1)}}\frac{A_{k,\ell,s}t_k^{-n}}{a^{s-1}(\omega_n^j)n^{\alpha_k s+\ell-\alpha_1+1}}+\omega_n^j a'(\omega_n^j)Q_{15}(j,n)+Q_{16}(j,n),\end{aligned}$$

where $Q_{16}(j,n)=O(\log^2 n/n^2)$ as $n\to\infty$, uniformly with respect to $j\in\mathcal{J}_n$. □

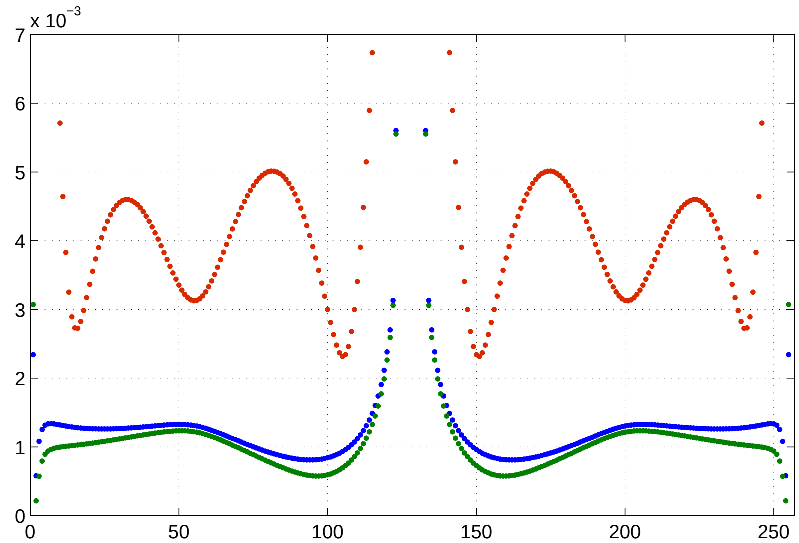

Figure 6. The absolute value of the difference between the eigenvalues of $T_{256}\big(t^{-1}(1-t)^{0.6}(1+t)^{0.9}\big)$ obtained with *Matlab* and formula (6.2). The red, blue, and green dots correspond to the approximations of (6.2) with 2, 3, and 4 terms, respectively.

6 Examples

In this Section we consider two particular situations for symbols with two and three singularities. In these situations we employ our formulas for $t_{\lambda_{j,n}}$ and $\lambda_{j,n}$, and with the aid of *Matlab*, we calculate the corresponding numerical errors.

Example 6.1. Consider the symbol $a(t)=t^{-1}(1-t)^{0.6}(1+t)^{0.9}$ with two singularities. In this case equations (1.2) and (1.3) become

$$t_{\lambda_{j,n}} = \omega_n^j n^{1.6/n}\left(1+\frac{1}{n}\log\left(\frac{a^2(\omega_n^j)}{A_{1,0,1}\omega_n^{2j}a'(\omega_n^j)}\right)-\frac{(-1)^n A_{2,0,1}}{A_{1,0,1}n^{1.3}}+R_2(j,n)\right) \tag{6.1}$$

and

$$\begin{aligned}\lambda_{j,n} = a(\omega_n^j)+1.6\omega_n^j a'(\omega_n^j)\frac{\log n}{n}+\frac{\omega_n^j a'(\omega_n^j)}{n}\log\left(\frac{a^2(\omega_n^j)}{A_{1,0,1}\omega_n^{2j}a'(\omega_n^j)}\right)\\ -\frac{(-1)^n A_{2,0,1}\omega_n^j a'(\omega_n^j)}{A_{1,0,1}n^{1.3}}+R_3(j,n),\end{aligned} \tag{6.2}$$

respectively. Here

$$A_{1,0,1}=2^{0.9}\sin(0.6\pi)\Gamma(1.6)/\pi,\quad A_{2,0,1}=2^{0.6}\sin(0.9\pi)\Gamma(1.9)/\pi,$$

and R_2, R_3 are $O(1/n^{1.6})$ as $n\to\infty$, uniformly with respect to $j\in\mathcal{J}_n$. Table 1 shows the data, see also Figures 2 and 6.

Example 6.2. Consider now the symbol

$$a(t)=t^{-1}(1-t)^{0.4}(1-t/e^{2i})^{0.6}(1-t/e^{4i})^{0.7}$$

n	256	512	1024	2048	4096
(6.1) with 1 term	1.1×10^{-2}	6.8×10^{-3}	3.3×10^{-3}	1.7×10^{-3}	8.4×10^{-4}
(6.1) with 2 terms	2.6×10^{-3}	7.9×10^{-4}	2.3×10^{-4}	7.1×10^{-5}	2.2×10^{-5}
(6.1) with 3 terms	2.5×10^{-3}	7.9×10^{-4}	2.2×10^{-4}	6.6×10^{-5}	1.9×10^{-5}
(6.2) with 2 term	1.4×10^{-2}	7.1×10^{-3}	3.5×10^{-3}	1.7×10^{-3}	8.5×10^{-4}
(6.2) with 3 terms	1.6×10^{-3}	5.8×10^{-4}	2.2×10^{-4}	7.5×10^{-5}	2.6×10^{-5}
(6.2) with 4 terms	1.4×10^{-3}	4.4×10^{-4}	1.8×10^{-4}	6.0×10^{-5}	2.0×10^{-5}

Table 1. The table shows the maximum error obtained with formulas (6.1) and (6.2) for the eigenvalues of the matrices $T_n\big(t^{-1}(1-t)^{0.6}(1+t)^{0.9}\big)$ for different values of n. The data was obtained by comparison with the solutions given by *Matlab*, taking into account only the 90% best approximated eigenvalues.

with three singularities. In this case equations (1.2) and (1.3) read

$$t_{\lambda_{j,n}} = \omega_n^j n^{1.4/n}\left(1+\frac{1}{n}\log\left(\frac{a^2(\omega_n^j)}{A_{1,0,1}\omega_n^{2j}a'(\omega_n^j)}\right)\right. \\ \left. -\frac{A_{2,0,1}e^{-2ni}}{A_{1,0,1}n^{1.2}}-\frac{A_{3,0,1}e^{-4ni}}{A_{1,0,1}n^{1.3}}+R_2(j,n)\right) \tag{6.3}$$

and

$$\lambda_{j,n} = a(\omega_n^j)+1.4\omega_n^j a'(\omega_n^j)\frac{\log n}{n}+\frac{\omega_n^j a'(\omega_n^j)}{n}\log\left(\frac{a^2(\omega_n^j)}{A_{1,0,1}\omega_n^{2j}a'(\omega_n^j)}\right) \\ -\frac{A_{2,0,1}e^{-2ni}\omega_n^j a'(\omega_n^j)}{A_{1,0,1}n^{1.2}}-\frac{A_{3,0,1}e^{-4ni}\omega_n^j a'(\omega_n^j)}{A_{1,0,1}n^{1.3}}+R_3(j,n) \tag{6.4}$$

respectively. Here

$$A_{1,0,1} = \sin(0.4\pi)\Gamma(1.4)(1-e^{-2i})^{0.6}(1-e^{-4i})^{0.7}/\pi,$$
$$A_{2,0,1} = \sin(0.6\pi)\Gamma(1.6)(1-e^{2i})^{0.4}(1-e^{-2i})^{0.7}/(\pi e^{4i}),$$
$$A_{3,0,1} = \sin(0.7\pi)\Gamma(1.7)(1-e^{4i})^{0.4}(1-e^{2i})^{0.6}/(\pi e^{8i}),$$

and R_2, R_3 are $O(1/n^{1.4})$ as $n\to\infty$, uniformly with respect to $j\in\mathcal{J}_n$. Table 2 shows the data, see also Figure 2.

n	256	512	1024	2048	4096
(6.3) with 1 term	2.5×10^{-2}	1.1×10^{-2}	6.2×10^{-3}	3.1×10^{-3}	1.6×10^{-3}
(6.3) with 2 terms	1.0×10^{-2}	3.0×10^{-3}	9.0×10^{-4}	2.8×10^{-4}	9.5×10^{-5}
(6.3) with 4 terms	7.8×10^{-3}	2.4×10^{-3}	6.8×10^{-4}	2.3×10^{-4}	7.8×10^{-5}
(6.4) with 2 terms	2.6×10^{-2}	1.2×10^{-2}	6.4×10^{-3}	3.2×10^{-3}	1.6×10^{-3}
(6.4) with 3 terms	9.2×10^{-3}	2.0×10^{-3}	6.3×10^{-4}	2.1×10^{-4}	7.8×10^{-5}
(6.4) with 5 terms	5.7×10^{-3}	1.8×10^{-3}	5.2×10^{-4}	1.9×10^{-4}	7.0×10^{-5}

Table 2. The table shows the maximum error obtained with formulas (6.3) and (6.4) for the eigenvalues of the matrices $T_n\big(t^{-1}(1-t/e^{2i})^{0.4}(1-t/e^{4i})^{0.6}(1-t/e^{6i})^{0.7}\big)$ for different values of n. The data was obtained by comparison with the solutions given by *Matlab*, taking into account only the 90% best approximated eigenvalues.

Tables 1 and 2 reveal that the maximum error of (1.2) with one term is reduced by nearly $n/80$ times when considering the second term; see also Figure 6.

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